

Worked solutions

Lesson 9 • Further induction III: induction in context

Warm-up • Part A

- (a) If $a_1 = 2$ and $a_{n+1} = 3a_n - 2$, find a_2, a_3, a_4 .

$a_2 = 4, a_3 = 10, a_4 = 28$. **A pattern:** $a_n = 3^{n-1} + 1$.

Apply the rule repeatedly: $3(2) - 2 = 4, 3(4) - 2 = 10, 3(10) - 2 = 28$. These equal $3^1 + 1, 3^2 + 1, 3^3 + 1$.

- (b) Differentiate xe^x , then differentiate the answer again.

$(x+1)e^x$, then $(x+2)e^x$. **A pattern:** $(x+n)e^x$.

Product rule: $e^x + xe^x = (x+1)e^x$. Again: $e^x + (x+1)e^x = (x+2)e^x$.

- (c) Write $\sin 2A$ and $\cos 2A$ in terms of $\sin A$ and $\cos A$.

$\sin 2A = 2 \sin A \cos A$ and $\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$.

These are the double angle formulas. The three forms of $\cos 2A$ come from $\sin^2 A + \cos^2 A = 1$.

- (d) Show that $1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$.

Put $A = \frac{\theta}{2}$ in $\cos 2A = 2 \cos^2 A - 1$: $\cos \theta = 2 \cos^2 \frac{\theta}{2} - 1$, so $1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$.

Parts (a), (b) and (d) are all used later in the lesson.

Example 1

Prove by induction that if $a_1 = 2$ and $a_{n+1} = 3a_n - 2$, then $a_n = 3^{n-1} + 1$ for all integers $n \geq 1$.

Let $P(n)$ be the statement $a_n = 3^{n-1} + 1$.

Step 1. $P(1)$: $3^0 + 1 = 2 = a_1$, so $P(1)$ is true.

Step 2. Assume $P(k)$: $a_k = 3^{k-1} + 1$ for some integer $k \geq 1$.

Step 3.

$$a_{k+1} = 3a_k - 2 = 3(3^{k-1} + 1) - 2 = 3^k + 1.$$

So $P(k+1)$ is true when $P(k)$ is true.

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

The assumption is used inside the recursion: replace a_k in $a_{k+1} = 3a_k - 2$ by the assumed formula. This differs from series, where you add a term to the assumed sum.

Example 2

A sequence is defined by $u_1 = 1$ and $u_{n+1} = \frac{u_n}{1+u_n}$. Find u_2, u_3, u_4 , make a conjecture for u_n , and prove it by induction.

$$u_2 = \frac{1}{2}, u_3 = \frac{1}{3}, u_4 = \frac{1}{4}. \text{ Conjecture: } u_n = \frac{1}{n}.$$

Step 1. $u_1 = 1 = \frac{1}{1}$.

Step 2. Assume $u_k = \frac{1}{k}$ for some integer $k \geq 1$.

Step 3.

$$u_{k+1} = \frac{u_k}{1+u_k} = \frac{1/k}{1+1/k} = \frac{1/k}{(k+1)/k} = \frac{1}{k+1}.$$

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

Conjecture first from the pattern, then prove it: this is the reasoning and communication the syllabus asks for. In Step 3, multiply top and bottom of the complex fraction by k .

HSC practice • A nested square root

The sequence a_n is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. Prove by induction that $a_n = 2 \cos \left(\frac{\pi}{2^{n+1}} \right)$ for all integers $n \geq 1$. You may use $\sqrt{2 + 2 \cos \theta} = 2 \cos \frac{\theta}{2}$ for $0 \leq \theta \leq \pi$ (warm-up (d)).

Step 1. $P(1)$: $2 \cos \frac{\pi}{4} = 2 \times \frac{\sqrt{2}}{2} = \sqrt{2} = a_1$, so $P(1)$ is true.

Step 2. Assume $a_k = 2 \cos \frac{\pi}{2^{k+1}}$ for some integer $k \geq 1$.

Step 3.

$$a_{k+1} = \sqrt{2 + 2 \cos \frac{\pi}{2^{k+1}}} = \sqrt{4 \cos^2 \frac{\pi}{2^{k+2}}} = 2 \cos \frac{\pi}{2^{k+2}},$$

taking the positive root since $0 < \frac{\pi}{2^{k+2}} < \frac{\pi}{2}$, so the cosine is positive.

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

Use the given identity with $\theta = \frac{\pi}{2^{k+1}}$, which lies in $[0, \pi]$. Then $\frac{\theta}{2} = \frac{\pi}{2^{k+2}}$. The positive square root must be justified: $\sqrt{4 \cos^2 \phi} = 2 |\cos \phi| = 2 \cos \phi$ only because $\cos \phi > 0$. (Wording adapted from the paper.)

Example 3

(a) Prove by induction that $\frac{d^n}{dx^n} (xe^x) = (x+n)e^x$ for all integers $n \geq 1$.

Step 1. $P(1)$: $\frac{d}{dx} (xe^x) = e^x + xe^x = (x+1)e^x$.

Step 2. Assume $\frac{d^k}{dx^k} (xe^x) = (x+k)e^x$ for some integer $k \geq 1$.

Step 3.

$$\frac{d^{k+1}}{dx^{k+1}} (xe^x) = \frac{d}{dx} [(x+k)e^x] = e^x + (x+k)e^x = (x+k+1)e^x.$$

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

The $(k+1)$ th derivative is the derivative of the k th derivative. Use the product rule on $(x+k)e^x$, treating k as a constant.

- (b) Prove by induction that, for $\sin x \neq 0$, $\cos x \cos 2x \cos 4x \cdots \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \sin x}$.

Step 1. $P(1)$: $\frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x$.

Step 2. Assume the product up to $\cos(2^{k-1}x)$ equals $\frac{\sin(2^k x)}{2^k \sin x}$ for some integer $k \geq 1$.

Step 3. Multiply by $\cos(2^k x)$:

$$\frac{\sin(2^k x) \cos(2^k x)}{2^k \sin x} = \frac{\frac{1}{2} \sin(2^{k+1} x)}{2^k \sin x} = \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}.$$

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

The $(k + 1)$ th product is the k th product times one new factor, $\cos(2^k x)$. The double angle formula with $A = 2^k x$ gives $\sin A \cos A = \frac{1}{2} \sin 2A$.

HSC practice • A trigonometric sum

(i) Show that $\cot x - \cot 2x = \csc 2x$.

(ii) Hence prove by induction that for all integers $n \geq 1$,

$$\sum_{r=1}^n \csc(2^r x) = \cot x - \cot(2^n x).$$

(i)

$$\cot x - \cot 2x = \frac{\cos x}{\sin x} - \frac{\cos 2x}{2 \sin x \cos x} = \frac{2 \cos^2 x - \cos 2x}{2 \sin x \cos x} = \frac{1}{\sin 2x} = \csc 2x.$$

(ii) **Step 1.** $P(1)$: $\csc 2x = \cot x - \cot 2x$ by (i).

Step 2. Assume $\sum_{r=1}^k \csc(2^r x) = \cot x - \cot(2^k x)$ for some integer $k \geq 1$.

Step 3. Add $\csc(2^{k+1}x) = \cot(2^k x) - \cot(2^{k+1}x)$, which is (i) with x replaced by $2^k x$:

$$\sum_{r=1}^{k+1} \csc(2^r x) = \cot x - \cot(2^k x) + \cot(2^k x) - \cot(2^{k+1}x) = \cot x - \cot(2^{k+1}x).$$

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

In (i), $\cos 2x = 2 \cos^2 x - 1$ makes the numerator 1, and $2 \sin x \cos x = \sin 2x$. The key idea in (ii) is “(i) with x replaced by $2^k x$ ”, which students often miss. (Cosec is written $\csc$ here; both are fine. Wording adapted from the paper.)

Spot the flaw • What's missing?

What is missing from this proof, and why would a marker deduct for it?

Nothing written is false, but three things are missing.

1. The step $2 + 2 \cos 2\theta = 4 \cos^2 \theta$ (from $\cos 2\theta = 2 \cos^2 \theta - 1$, with $\theta = \frac{\pi}{2^{k+1}}$) is not shown.

2. $\sqrt{4 \cos^2 \theta} = 2|\cos \theta|$, so we must justify $\cos \frac{\pi}{2^{k+1}} > 0$. This is true because $0 < \frac{\pi}{2^{k+1}} \leq \frac{\pi}{4} < \frac{\pi}{2}$ for $k \geq 1$.

3. There is no conclusion naming $P(k + 1)$ and the start value: “ $P(1)$ is true, and $P(k) \Rightarrow P(k + 1)$, so $P(n)$ is true for all $n \geq 1$ by induction.”

Compare the 2022 HSC question earlier in this lesson, which has the same structure with $a_1 = \sqrt{2}$. In an induction proof every step must be justified, in particular the choice of the positive square root, and the conclusion must be stated.

Warm-up • Part B

- (a) Evaluate $\binom{5}{2} + \binom{5}{3}$ and $\binom{6}{3}$. What do you notice?

$$10 + 10 = 20 = \binom{6}{3}.$$

This is Pascal's identity: $\binom{5}{2} + \binom{5}{3} = \binom{6}{3}$.

- (b) Expand $(1 + x)^4$ using Pascal's triangle.

$$1 + 4x + 6x^2 + 4x^3 + x^4.$$

The row of Pascal's triangle 1, 4, 6, 4, 1 gives the coefficients.

- (c) Simplify $\ln \left(1 - \frac{1}{r^2} \right)$ as a single log of a product of two fractions.

$$\ln \left[\frac{r-1}{r} \times \frac{r+1}{r} \right].$$

$1 - \frac{1}{r^2} = \frac{r^2 - 1}{r^2} = \frac{(r-1)(r+1)}{r^2}$, which is the product $\frac{r-1}{r} \times \frac{r+1}{r}$. Equivalently $\ln \frac{r-1}{r} + \ln \frac{r+1}{r}$.

- (d) How many diagonals does a hexagon have?

9.

Each of the 6 vertices joins to 3 non-adjacent vertices, and each diagonal is counted twice: $\frac{6 \times 3}{2} = 9$.

Example 4

- (a) Prove Pascal's identity using the factorial formula.

Using $\binom{n}{r} = \frac{n!}{r!(n-r)!}$:

$$\binom{k}{r-1} + \binom{k}{r} = \frac{k!}{(r-1)!(k-r+1)!} + \frac{k!}{r!(k-r)!} = \frac{k! [r + (k-r+1)]}{r!(k-r+1)!} = \frac{(k+1)!}{r!(k+1-r)!} = \binom{k+1}{r}.$$

Use the common denominator $r!(k-r+1)!$. The first fraction needs a factor r (since $r! = r(r-1)!$) and the second needs $(k-r+1)$ (since $(k-r+1)! = (k-r+1)(k-r)!$). The numerator becomes $k!(k+1)$.

(b) Prove the binomial theorem by induction.

Step 1. $P(1)$: $(1+x)^1 = 1+x = \binom{1}{0} + \binom{1}{1}x$.

Step 2. Assume $(1+x)^k = \sum_{r=0}^k \binom{k}{r} x^r$ for some integer $k \geq 1$.

Step 3.

$$(1+x)^{k+1} = (1+x) \sum_{r=0}^k \binom{k}{r} x^r = \sum_{r=0}^k \binom{k}{r} x^r + \sum_{r=0}^k \binom{k}{r} x^{r+1}.$$

The coefficient of x^r for $1 \leq r \leq k$ is $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$ by (a). The constant term is $1 = \binom{k+1}{0}$

and the coefficient of x^{k+1} is $1 = \binom{k+1}{k+1}$. Hence $(1+x)^{k+1} = \sum_{r=0}^{k+1} \binom{k+1}{r} x^r$.

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

Write out a few terms rather than relying only on the sigma notation: the first sum gives the coefficients of x^r and the second sum, shifted along by one power, gives the coefficients of x^r from $\binom{k}{r-1}$. The binomial theorem was used in Extension 1; this is where it is proved.

Example 5

(a) Given $\ln(ab) = \ln a + \ln b$, prove by induction that for positive $a_1, \dots, a_n$: $\ln(a_1 a_2 \cdots a_n) = \sum_{i=1}^n \ln a_i$.

Step 1. $P(1)$: $\ln a_1 = \ln a_1$.

Step 2. Assume $\ln(a_1 \cdots a_k) = \sum_{i=1}^k \ln a_i$ for some integer $k \geq 1$.

Step 3.

$$\ln(a_1 \cdots a_k a_{k+1}) = \ln(a_1 \cdots a_k) + \ln a_{k+1} = \sum_{i=1}^k \ln a_i + \ln a_{k+1} = \sum_{i=1}^{k+1} \ln a_i.$$

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

The log law $\ln(ab) = \ln a + \ln b$ is applied with $a = a_1 \cdots a_k$ and $b = a_{k+1}$. Then the assumption replaces $\ln(a_1 \cdots a_k)$.

(b) Prove by induction that for all integers $n \geq 2$: $\sum_{r=2}^n \ln\left(1 - \frac{1}{r^2}\right) = \ln\left(\frac{n+1}{2n}\right)$.

Step 1. $P(2)$: $\ln\left(1 - \frac{1}{4}\right) = \ln \frac{3}{4} = \ln \frac{3}{2 \times 2}$, true.

Step 2. Assume the result for some integer $k \geq 2$.

Step 3.

$$\sum_{r=2}^{k+1} \ln\left(1 - \frac{1}{r^2}\right) = \ln \frac{k+1}{2k} + \ln\left(1 - \frac{1}{(k+1)^2}\right) = \ln \left[\frac{k+1}{2k} \times \frac{k(k+2)}{(k+1)^2} \right] = \ln \frac{k+2}{2(k+1)}.$$

Step 4. True for $n = 2$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 2$ by induction.

The starting value is $n = 2$. Use $1 - \frac{1}{(k+1)^2} = \frac{(k+1)^2 - 1}{(k+1)^2} = \frac{k(k+2)}{(k+1)^2}$. A non-induction method is to telescope using warm-up (c).

Example 6

Prove by induction that a convex polygon with n sides has $\frac{n(n-3)}{2}$ diagonals, for all integers $n \geq 3$.

Step 1. $P(3)$: a triangle has 0 diagonals and $\frac{3 \times 0}{2} = 0$.

Step 2. Assume a convex k -gon has $\frac{k(k-3)}{2}$ diagonals, for some integer $k \geq 3$.

Step 3. Add a vertex V outside the edge AB to make a convex $(k+1)$ -gon. The new diagonals are V joined to the $k-2$ vertices other than A and B , and the old side AB is now a diagonal: $k-1$ new diagonals.

$$\frac{k(k-3)}{2} + k - 1 = \frac{k^2 - k - 2}{2} = \frac{(k+1)(k-2)}{2} = \frac{(k+1)((k+1)-3)}{2}.$$

Step 4. True for $n = 3$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 3$ by induction.

In a geometry proof, Step 3 must say in words what the $(k+1)$ th step adds. Draw a hexagon becoming a heptagon: V is adjacent only to A and B , so it makes diagonals to the other $k-2$ vertices, and AB stops being a side.

Exit ticket

A sequence is defined by $a_1 = 5$ and $a_{n+1} = 2a_n - 3$. Prove by induction that $a_n = 2^n + 3$ for all integers $n \geq 1$.

Step 1. $P(1)$: $2 + 3 = 5 = a_1$, so $P(1)$ is true.

Step 2. Assume $a_k = 2^k + 3$ for some integer $k \geq 1$.

Step 3. $a_{k+1} = 2a_k - 3 = 2(2^k + 3) - 3 = 2^{k+1} + 3$. So $P(k+1)$ is true when $P(k)$ is true.

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

Substitute the assumption into the recursion. Do not use the formula for a_{k+1} in Step 3, as that is what you are proving. Check: $a_2 = 7 = 2^2 + 3$.

Practice worksheet

Recursive sequences

1. A sequence has $a_1 = 3$ and $a_{n+1} = 2a_n - 1$. Prove by induction that $a_n = 2^n + 1$ for all integers $n \geq 1$.

Step 1. $n = 1$: $2 + 1 = 3 = a_1$.

Step 2. Assume $a_k = 2^k + 1$ for some integer $k \geq 1$.

Step 3. $a_{k+1} = 2a_k - 1 = 2(2^k + 1) - 1 = 2^{k+1} + 1$.

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

2. A sequence has $u_1 = 1$ and $u_{n+1} = u_n + 2n + 1$. Prove by induction that $u_n = n^2$ for all integers $n \geq 1$.

Step 1. $n = 1$: $1^2 = 1 = u_1$.

Step 2. Assume $u_k = k^2$ for some integer $k \geq 1$.

Step 3. $u_{k+1} = u_k + 2k + 1 = k^2 + 2k + 1 = (k+1)^2$.

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

Calculus

3. Prove by induction that $\frac{d^n}{dx^n}(xe^{-x}) = (-1)^n(x-n)e^{-x}$ for all integers $n \geq 1$.

Step 1. $n = 1$: $\frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x} = (1-x)e^{-x} = (-1)^1(x-1)e^{-x}$.

Step 2. Assume the k th derivative is $(-1)^k(x-k)e^{-x}$ for some integer $k \geq 1$.

Step 3. Differentiate: $(-1)^k[e^{-x} - (x-k)e^{-x}] = (-1)^k(1+k-x)e^{-x} = (-1)^{k+1}(x-(k+1))e^{-x}$.

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

Binomial theorem and logarithms

4. Use Pascal's identity to prove by induction that $\sum_{r=0}^n \binom{n}{r} = 2^n$ for all integers $n \geq 1$.

Step 1. $n = 1$: $\binom{1}{0} + \binom{1}{1} = 2 = 2^1$.

Step 2. Assume $\sum_{r=0}^k \binom{k}{r} = 2^k$ for some integer $k \geq 1$.

Step 3.

$$\sum_{r=0}^{k+1} \binom{k+1}{r} = 1 + \sum_{r=1}^k \left[\binom{k}{r-1} + \binom{k}{r} \right] + 1 = \sum_{r=0}^k \binom{k}{r} + \sum_{r=0}^k \binom{k}{r} = 2^k + 2^k = 2^{k+1}.$$

(Check: $\sum_{r=1}^k \binom{k}{r-1} = \sum_{r=0}^{k-1} \binom{k}{r} = 2^k - 1$ and $\sum_{r=1}^k \binom{k}{r} = 2^k - 1$, so the total is $1 + (2^k - 1) + (2^k - 1) + 1 = 2^{k+1}$.)

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.

5. Prove by induction that $\sum_{r=2}^n \ln \left(1 - \frac{1}{r} \right) = -\ln n$ for all integers $n \geq 2$.

Step 1. $n = 2$: $\ln \frac{1}{2} = -\ln 2$.

Step 2. Assume the result for some integer $k \geq 2$.

Step 3.

$$\sum_{r=2}^{k+1} \ln \left(1 - \frac{1}{r} \right) = -\ln k + \ln \frac{k}{k+1} = -\ln k + \ln k - \ln(k+1) = -\ln(k+1).$$

Step 4. True for $n = 2$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 2$ by induction.

Geometry

6. Prove by induction that the interior angles of a convex polygon with n sides sum to $(n - 2) \times 180^\circ$, for all integers $n \geq 3$.

Step 1. $n = 3$: a triangle has angle sum $180^\circ = (3 - 2) \times 180^\circ$.

Step 2. Assume a convex k -gon has angle sum $(k - 2) \times 180^\circ$, for some integer $k \geq 3$.

Step 3. In a convex $(k + 1)$ -gon $A_1 A_2 \cdots A_{k+1}$, draw the diagonal $A_1 A_k$. It splits the polygon into the k -gon $A_1 A_2 \cdots A_k$ and the triangle $A_1 A_k A_{k+1}$. The angles of the $(k + 1)$ -gon are the angles of the k -gon plus the angles of the triangle, so the sum is $(k - 2) \times 180^\circ + 180^\circ = ((k + 1) - 2) \times 180^\circ$.

Step 4. True for $n = 3$, and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 3$ by induction.

Adapted HSC-style

7. A sequence is defined by $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 3}$ for $n \geq 1$. (a) Prove by induction that $0 < a_n < 3$ for all integers $n \geq 1$. (b) Prove by induction that $a_{n+1} > a_n$ for all integers $n \geq 1$.

(a) **Step 1.** $a_1 = 1 \in (0, 3)$. **Step 2.** Assume $0 < a_k < 3$. **Step 3.** Then $3 < 2a_k + 3 < 9$, so $\sqrt{3} < a_{k+1} < 3$, hence $0 < a_{k+1} < 3$. **Step 4.** By induction, true for all $n \geq 1$.

(b) **Step 1.** $a_2 = \sqrt{5} > 1 = a_1$. **Step 2.** Assume $a_{k+1} > a_k$ for some integer $k \geq 1$. **Step 3.** Then $2a_{k+1} + 3 > 2a_k + 3$, and taking square roots (both sides positive), $a_{k+2} = \sqrt{2a_{k+1} + 3} > \sqrt{2a_k + 3} = a_{k+1}$. **Step 4.** By induction, $a_{n+1} > a_n$ for all $n \geq 1$.