

# Worked solutions

Lesson 8 • Further induction II: inequalities

## Warm-up

- (a) For which positive integers  $n$  is  $n^2 > 2n + 1$ ?

$$n \geq 3.$$

$n^2 > 2n + 1 \iff (n - 1)^2 > 2$ , which needs  $n - 1 \geq 2$  for a positive integer, so  $n \geq 3$ . Check:  
 $n = 2$  gives  $4 > 5$ , false;  $n = 3$  gives  $9 > 7$ , true.

- (b) Show that  $2k^2 - (k + 1)^2 = (k - 1)^2 - 2$ .

**Both sides equal  $k^2 - 2k - 1$ .**

LHS:  $2k^2 - (k^2 + 2k + 1) = k^2 - 2k - 1$ . RHS:  $(k^2 - 2k + 1) - 2 = k^2 - 2k - 1$ .

- (c) If  $A > B$  and  $B \geq C$ , what can you conclude?

$$A > C.$$

A strict inequality followed by a non-strict one still gives a strict result. This is the idea behind the chain layout used in this lesson.

- (d) Test  $2^n > n^2$  for  $n = 1, 2, 3, 4, 5, 6$ .

**True for  $n = 1$ , false for  $n = 2, 3, 4$ , true for  $n = 5, 6$ . So claim  $n \geq 5$ .**

$2 > 1$  yes;  $4 = 4$  no;  $8 < 9$  no;  $16 = 16$  no;  $32 > 25$  yes;  $64 > 36$  yes. The claim holds from  $n = 5$  on, so the first value of  $n$  must be chosen with care.

## Example 1

Prove by induction that  $2^n > n^2$  for all integers  $n \geq 5$ .

**Let  $P(n)$ :  $2^n > n^2$ .**

**Step 1.  $P(5)$ :  $2^5 = 32 > 25 = 5^2$ .  $\therefore P(5)$  is true.**

**Step 2. Assume  $2^k > k^2$  for some integer  $k \geq 5$ .**

**Step 3. Need  $2^{k+1} > (k + 1)^2$ .**

$$2^{k+1} = 2 \cdot 2^k > 2k^2 \geq (k + 1)^2.$$

**The last step holds because  $2k^2 - (k + 1)^2 = (k - 1)^2 - 2 \geq 16 - 2 > 0$  for  $k \geq 5$ .  $\therefore P(k + 1)$  is true whenever  $P(k)$  is true.**

**Step 4. By mathematical induction,  $2^n > n^2$  for all integers  $n \geq 5$ .**

The assumption gives the first link. The “gap”  $2k^2 \geq (k + 1)^2$  is proved separately (Warm-up (b)), and it is where  $k \geq 5$  is used. Without a large enough  $k$  the gap fails, for example at  $k = 2$ .

## Example 2 • Factorials and powers

(a) Prove by induction that  $n! > 2^n$  for all integers  $n \geq 4$ .

**Step 1.**  $n = 4$ :  $4! = 24 > 16 = 2^4$ .

**Step 2.** Assume  $k! > 2^k$  for some integer  $k \geq 4$ .

**Step 3.** Need  $(k+1)! > 2^{k+1}$ .

$$(k+1)! = (k+1)k! > (k+1)2^k \geq 2 \cdot 2^k = 2^{k+1},$$

since  $k+1 \geq 2$ .

**Step 4.** True for  $n = 4$  and true for  $n = k \Rightarrow n = k+1$ , so true for all integers  $n \geq 4$  by induction.

Write  $(k+1)! = (k+1)k!$  so the assumption can be used. The factor  $(k+1)$  is at least 2, which turns  $2^k$  into  $2^{k+1}$ . Check:  $n = 3$  gives  $6 > 8$ , false, which is why the base case is  $n = 4$ .

(b) Prove by induction that  $3^n \geq 1 + 2n$  for all integers  $n \geq 1$ .

**Step 1.**  $n = 1$ :  $3^1 = 3 \geq 1 + 2 = 3$ .

**Step 2.** Assume  $3^k \geq 1 + 2k$  for some integer  $k \geq 1$ .

**Step 3.** Need  $3^{k+1} \geq 1 + 2(k+1) = 2k + 3$ .

$$3^{k+1} = 3 \cdot 3^k \geq 3(1 + 2k) = 3 + 6k \geq 3 + 2k = 1 + 2(k+1),$$

since  $4k \geq 0$ .

**Step 4.** True for  $n = 1$  and true for  $n = k \Rightarrow n = k+1$ , so true for all integers  $n \geq 1$  by induction.

Multiply the assumption by 3 (positive, so the inequality direction is preserved). The gap is  $3 + 6k - (3 + 2k) = 4k \geq 0$ . The base case gives equality, so the inequality is  $\geq$  rather than  $>$ .

## 2023 HSC Q13(b)

(i) Show that  $k^2 - 2k - 3 \geq 0$  for all integers  $k \geq 3$ .

$$k^2 - 2k - 3 = (k-3)(k+1) \geq 0 \text{ for } k \geq 3.$$

For  $k \geq 3$  both factors are non-negative:  $k-3 \geq 0$  and  $k+1 > 0$ .

(ii) Hence prove by induction that  $2^n \geq n^2 - 2$  for all integers  $n \geq 3$ .

**Step 1.**  $P(3)$ :  $8 \geq 9 - 2 = 7$ . True.

**Step 2.** Assume  $2^k \geq k^2 - 2$  for some integer  $k \geq 3$ .

**Step 3.** Then  $2^{k+1} = 2 \cdot 2^k \geq 2k^2 - 4$ . Need  $2^{k+1} \geq (k+1)^2 - 2 = k^2 + 2k - 1$ . Now

$$2k^2 - 4 - (k^2 + 2k - 1) = k^2 - 2k - 3 \geq 0$$

by (i), so  $2^{k+1} \geq 2k^2 - 4 \geq (k+1)^2 - 2$ .

**Step 4.** True for  $n = 3$  and true for  $n = k \Rightarrow n = k+1$ , so true for all integers  $n \geq 3$  by induction.

Part (i) is exactly the gap between  $2k^2 - 4$  and the target  $(k+1)^2 - 2$ . Look for the given result inside the inductive step.

## Question A

Prove by induction that  $2^n < \sqrt{n!}$  for all integers  $n \geq 9$ .

**Step 1.**  $P(9)$ :  $2^9 = 512$  and  $\sqrt{9!} = \sqrt{362\,880} \approx 602.4$ , so  $512 < 602.4$ . **True.**

**Step 2.** Assume  $2^k < \sqrt{k!}$  for some integer  $k \geq 9$ .

**Step 3.**

$$2^{k+1} = 2 \cdot 2^k < 2\sqrt{k!} < \sqrt{k+1} \sqrt{k!} = \sqrt{(k+1)!},$$

since  $\sqrt{k+1} \geq \sqrt{10} > 2$ .

**Step 4.** True for  $n = 9$  and true for  $n = k \Rightarrow n = k + 1$ , so true for all integers  $n \geq 9$  by induction.

Split  $\sqrt{(k+1)!} = \sqrt{k+1} \sqrt{k!}$  so the assumption can be used, then compare the factor  $\sqrt{k+1}$  with 2. This is where  $k \geq 9$  is needed. For the base case,  $602^2 = 362\,404 < 9! = 362\,880$ , so  $\sqrt{9!} > 602 > 512$ .

## Question B

Prove by induction that  $2^n + 3^n < 5^n$  for all integers  $n \geq 2$ .

**Step 1.**  $P(2)$ :  $4 + 9 = 13 < 25$ . **True.**

**Step 2.** Assume  $2^k + 3^k < 5^k$  for some integer  $k \geq 2$ .

**Step 3.**

$$2^{k+1} + 3^{k+1} = 2 \cdot 2^k + 3 \cdot 3^k < 3(2^k + 3^k) < 3 \cdot 5^k < 5^{k+1}.$$

**Step 4.** True for  $n = 2$  and true for  $n = k \Rightarrow n = k + 1$ , so true for all integers  $n \geq 2$  by induction.

Replace both multipliers by the larger one, 3, so the bracket  $2^k + 3^k$  from the assumption appears. Then  $3 \cdot 5^k < 5 \cdot 5^k$ . A non-induction proof expands  $(2 + 3)^n$  by the binomial theorem (Lesson 3).

## HSC practice • A binomial coefficient

Prove by induction that  $\binom{2n}{n} < 2^{2n-2}$  for all integers  $n \geq 5$ .

Hint:  $\binom{2k+2}{k+1} = \frac{2(2k+1)}{k+1} \binom{2k}{k}$ .

**Step 1.**  $P(5)$ :  $\binom{10}{5} = 252 < 2^8 = 256$ . **True.**

**Step 2. Assume**  $\binom{2k}{k} < 2^{2k-2}$  **for some integer**  $k \geq 5$ .

**Step 3.**

$$\binom{2k+2}{k+1} = \frac{2(2k+1)}{k+1} \binom{2k}{k} < 4 \binom{2k}{k} < 4 \cdot 2^{2k-2} = 2^{2(k+1)-2},$$

since  $\frac{2k+1}{k+1} < 2$ .

**Step 4. True for**  $n = 5$  **and true for**  $n = k \Rightarrow n = k + 1$ , **so true for all integers**  $n \geq 5$  **by induction.**

The hint comes from  $\binom{2k+2}{k+1} = \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \binom{2k}{k}$  using the factorial formula; prove it first if it is not given. The ratio  $\frac{2(2k+1)}{k+1} = 4 - \frac{2}{k+1}$  is less than 4, and  $\binom{2k}{k} > 0$  keeps the inequality direction.

## HSC practice • A sum

Prove by induction that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}$$

for all integers  $n \geq 2$ .

**Step 1.**  $P(2)$ :  $\frac{1}{4} < \frac{1}{2}$ . **True.**

**Step 2. Assume**  $\sum_{r=2}^k \frac{1}{r^2} < \frac{k-1}{k}$  **for some integer**  $k \geq 2$ .

**Step 3.**  $\sum_{r=2}^{k+1} \frac{1}{r^2} < \frac{k-1}{k} + \frac{1}{(k+1)^2}$ . **Need this to be at most**  $\frac{k}{k+1}$ . **Now**

$$\frac{k}{k+1} - \frac{k-1}{k} = \frac{1}{k(k+1)} > \frac{1}{(k+1)^2},$$

**so**  $\frac{k-1}{k} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$ .

**Step 4. True for**  $n = 2$  **and true for**  $n = k \Rightarrow n = k + 1$ , **so true for all integers**  $n \geq 2$  **by induction.**

Add the next term to the assumption, then compare with the target by looking at the difference. Since  $k(k+1) < (k+1)^2$ , we have  $\frac{1}{k(k+1)} > \frac{1}{(k+1)^2}$ . Non-induction alternative:  $\frac{1}{r^2} < \frac{1}{r(r-1)} = \frac{1}{r-1} - \frac{1}{r}$ , which telescopes.

### Example 3 • A sum with square roots

Prove by induction that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

for all integers  $n \geq 1$ .

**Step 1.  $n = 1$ : LHS**  $= 1 \geq 1 = \sqrt{1}$ .

**Step 2. Assume**  $S_k \geq \sqrt{k}$  **for some integer**  $k \geq 1$ .

**Step 3. Need**  $S_{k+1} \geq \sqrt{k+1}$ .

$$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

**Now**  $\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$  **because, multiplying by**  $\sqrt{k+1} > 0$ , **this is equivalent to**  
 $\sqrt{k(k+1)} + 1 \geq k+1$ , **that is**  $\sqrt{k^2+k} \geq k$ , **which is true.**

**Step 4. True for**  $n = 1$  **and true for**  $n = k \Rightarrow n = k+1$ , **so true for all integers**  $n \geq 1$  **by induction.**

The gap inequality is a small separate problem. It is safe to multiply by  $\sqrt{k+1}$  because it is positive, and every step is reversible, so the gap inequality really does follow from the true statement  $\sqrt{k^2+k} \geq k$  (as  $k^2+k \geq k^2$ ).

### Spot the flaw • An induction inequality

**Task** Find the flaw, explain why it matters, then write a correct version.

**Flaw: line 3 is not justified.**  $2k^2 - (k+1)^2 = (k-1)^2 - 2$ , which is  $\geq 0$  only for  $k \geq 3$  ( $k = 1$ :  $2 < 4$ ;  $k = 2$ :  $8 < 9$ ). The claim is also false for  $n = 2, 3, 4$  ( $4 = 4$ ,  $8 < 9$ ,  $16 = 16$ ).

**Correct version: Example 1. Start at**  $n = 5$  ( $32 > 25$ ) **and prove the gap step for**  $k \geq 5$ .

Prove the “gap” inequality for every  $k$  the step needs, and test small cases before choosing the first value.

### Exit ticket

(a) Find the smallest positive integer  $n$  for which it is true (and so the correct base case).

$$n = 3.$$

$$n = 1: 4 > 13 \text{ false. } n = 2: 16 > 16 \text{ false. } n = 3: 64 > 19 \text{ true.}$$

(b) Assume  $4^k > 3k + 10$  for some integer  $k \geq 3$ . Write only the inductive step (Step 3) as a chain of inequalities.

$$4^{k+1} = 4 \cdot 4^k > 4(3k + 10) = 12k + 40 > 3k + 13 = 3(k+1) + 10.$$

The gap is  $12k + 40 - (3k + 13) = 9k + 27 > 0$ . Each link is justified: the first by the assumption, the second because  $9k + 27 > 0$ .

## Practice worksheet

### Choosing the first value

1. Find the smallest positive integer  $n$  from which each inequality is true for all larger  $n$  as well, by testing small values.

(a)  $2^n > 2n + 1$

(b)  $n! > n^2$

(c)  $2^n > n^2$

**(a)  $n = 3$  ( $2 < 3$ ,  $4 < 5$ ,  $8 > 7$ ). (b)  $n = 4$  ( $1 = 1$ ,  $2 < 4$ ,  $6 < 9$ ,  $24 > 16$ ). (c)  $n = 5$  ( $2 > 1$  but  $4 = 4$ ,  $8 < 9$ ,  $16 = 16$ , then  $32 > 25$ ).**

### Induction with inequalities

2. Prove by induction that  $2^n > 2n + 1$  for all integers  $n \geq 3$ .

**Step 1.  $n = 3$ :  $8 > 7$  true.**

**Step 2. Assume  $2^k > 2k + 1$  for some integer  $k \geq 3$ .**

**Step 3.  $2^{k+1} = 2 \cdot 2^k > 2(2k + 1) = 4k + 2 \geq 2k + 3 = 2(k + 1) + 1$ , since  $2k \geq 1$ .**

**Step 4. True for  $n = 3$ , and true for  $n = k \Rightarrow n = k + 1$ , so true for all integers  $n \geq 3$  by induction.**

3. Prove by induction that  $4^n \geq 1 + 3n$  for all integers  $n \geq 1$ .

**Step 1.  $n = 1$ :  $4 \geq 4$  true.**

**Step 2. Assume  $4^k \geq 1 + 3k$  for some integer  $k \geq 1$ .**

**Step 3.  $4^{k+1} = 4 \cdot 4^k \geq 4(1 + 3k) = 4 + 12k \geq 4 + 3k = 1 + 3(k + 1)$ , since  $9k \geq 0$ .**

**Step 4. True for  $n = 1$ , and true for  $n = k \Rightarrow n = k + 1$ , so true for all integers  $n \geq 1$  by induction.**

4. Prove by induction that  $n! \geq 2^{n-1}$  for all integers  $n \geq 1$ .

**Step 1.  $n = 1$ :  $1! = 1 \geq 2^0 = 1$  true.**

**Step 2. Assume  $k! \geq 2^{k-1}$  for some integer  $k \geq 1$ .**

**Step 3.  $(k + 1)! = (k + 1)k! \geq (k + 1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k = 2^{(k+1)-1}$ , since  $k + 1 \geq 2$ .**

**Step 4. True for  $n = 1$ , and true for  $n = k \Rightarrow n = k + 1$ , so true for all integers  $n \geq 1$  by induction.**

5. Prove by induction that  $(1 + x)^n \geq 1 + nx$  for all integers  $n \geq 1$ , where  $x > -1$  is a fixed real number.

**Step 1.  $n = 1$ :  $1 + x \geq 1 + x$  true.**

**Step 2. Assume  $(1 + x)^k \geq 1 + kx$  for some integer  $k \geq 1$ .**

**Step 3. Since  $1 + x > 0$ ,  $(1 + x)^{k+1} = (1 + x)(1 + x)^k \geq (1 + x)(1 + kx) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$ , because  $kx^2 \geq 0$ .**

**Step 4. True for  $n = 1$ , and true for  $n = k \Rightarrow n = k + 1$ , so true for all integers  $n \geq 1$  by induction.**

6. Prove by induction that  $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{1}{2}$  for all integers  $n \geq 1$ .

**Step 1.**  $n = 1$ : LHS =  $\frac{1}{2} \geq \frac{1}{2}$  true.

**Step 2.** Let  $T_k = \frac{1}{k+1} + \cdots + \frac{1}{2k}$  and assume  $T_k \geq \frac{1}{2}$  for some integer  $k \geq 1$ .

**Step 3.**  $T_{k+1} = T_k - \frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2} = T_k + \frac{1}{2k+1} - \frac{1}{2k+2}$ , because  $\frac{1}{k+1} = \frac{2}{2k+2}$ . Then  $\frac{1}{2k+1} - \frac{1}{2k+2} = \frac{1}{(2k+1)(2k+2)} > 0$ , so  $T_{k+1} > T_k \geq \frac{1}{2}$ .

**Step 4.** True for  $n = 1$ , and true for  $n = k \Rightarrow n = k + 1$ , so true for all integers  $n \geq 1$  by induction.

### Adapted HSC-style

7. (a) Show that  $2k^3 - 3k^2 - 3k - 1 > 0$  for integers  $k \geq 4$ . (b) Hence prove by induction that  $3^n > n^3$  for all integers  $n \geq 4$ .

**(a)**  $2k^3 - 3k^2 - 3k - 1 = k^2(2k - 3) - 3k - 1 \geq 5k^2 - 3k - 1 = k(5k - 3) - 1 \geq 4 \times 17 - 1 > 0$  for  $k \geq 4$ . **(b) Step 1.**  $n = 4$ :  $81 > 64$  true.

**Step 2.** Assume  $3^k > k^3$  for some integer  $k \geq 4$ .

**Step 3.**  $3^{k+1} = 3 \cdot 3^k > 3k^3 = (k+1)^3 + (2k^3 - 3k^2 - 3k - 1) > (k+1)^3$  by part (a).

**Step 4.** True for  $n = 4$ , and true for  $n = k \Rightarrow n = k + 1$ , so true for all integers  $n \geq 4$  by induction.