

Worked solutions

Lesson 6 • Algebraic inequalities II

Warm-up

- (a) Prove $p^2 + q^2 \geq 2pq$.

$$p^2 + q^2 - 2pq = (p - q)^2 \geq 0.$$

Subtract the right-hand side and recognise a perfect square.

- (b) Expand $(\sqrt{a} - \sqrt{b})^2$ for $a, b \geq 0$.

$$a - 2\sqrt{ab} + b.$$

$(\sqrt{a})^2 = a$, $(\sqrt{b})^2 = b$ and $\sqrt{a}\sqrt{b} = \sqrt{ab}$ since $a, b \geq 0$. This will give AM-GM.

- (c) Compare $|-3 + 5|$ with $|-3| + |5|$.

$$|-3 + 5| = 2 \text{ and } |-3| + |5| = 8, \text{ so } 2 < 8.$$

The two sides are equal only when the numbers have the same sign. This is the idea of the triangle inequality.

- (d) Can a triangle have sides 3 cm, 4 cm and 8 cm?

No.

$3 + 4 = 7 < 8$: the two shorter sides together are too short to reach across the longest side.

Example 1 (a)

Prove the AM-GM inequality.

$$(\sqrt{a} - \sqrt{b})^2 \geq 0 \Rightarrow a + b - 2\sqrt{ab} \geq 0 \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}. \text{ Equality holds if and only if } \sqrt{a} = \sqrt{b}, \text{ that is } a = b.$$

This is the expansion from the warm-up. A square is zero only when its base is zero. (A geometric proof also works: in a semicircle of diameter $a + b$, the half-chord at the join has length $\sqrt{ab}$ and cannot exceed the radius $\frac{a+b}{2}$.)

Example 1 (b)

Prove that for $x > 0$, $x + \frac{9}{x} \geq 6$. When is there equality?

$$x + \frac{9}{x} \geq 2\sqrt{x \cdot \frac{9}{x}} = 2\sqrt{9} = 6. \text{ Equality holds when } x = \frac{9}{x}, \text{ so } x^2 = 9 \text{ and } x = 3.$$

Apply AM-GM to the two positive numbers x and $\frac{9}{x}$. Their product is the constant 9, which is what makes the bound useful.

HSC practice • Substituting into AM-GM

- (i) Show that $a + b \geq 2\sqrt{ab}$ for all $a, b \geq 0$.
 (ii) Hence show that $p^2 + 4q^2 \geq 4pq$ for all real p and q .

(i) $(\sqrt{a} - \sqrt{b})^2 \geq 0$, so $a + b \geq 2\sqrt{ab}$. **(ii)** Put $a = p^2, b = 4q^2$ (both ≥ 0): $p^2 + 4q^2 \geq 2\sqrt{4p^2q^2} = 4|pq| \geq 4pq$.

Substitution is the key skill: choose a and b so that the square roots simplify. Check: $(p - 2q)^2 \geq 0$ gives the same result directly. (The original part (i) is proved geometrically from a diagram; the wording here is adapted.)

Example 2 • Using AM-GM

- (a)** Prove that $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$ for $a, b > 0$.

$$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) = 2 + \frac{a}{b} + \frac{b}{a} \geq 2 + 2 = 4.$$

Expand, then use $\frac{a}{b} + \frac{b}{a} \geq 2$ from Lesson 5. Alternatively apply AM-GM to each bracket: $a + b \geq 2\sqrt{ab}$ and $\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$, then multiply.

- (b)** Prove that $(a + b)(b + c)(c + a) \geq 8abc$ for $a, b, c > 0$.

$a + b \geq 2\sqrt{ab}, b + c \geq 2\sqrt{bc}, c + a \geq 2\sqrt{ca}$. All sides are positive, so multiply: $(a + b)(b + c)(c + a) \geq 8\sqrt{a^2b^2c^2} = 8abc$.

Multiplying inequalities is valid when all the quantities are positive. The three square roots multiply to $\sqrt{(abc)^2} = abc$.

HSC practice • Building on AM-GM

- (i) Using $\sqrt{xy} \leq \frac{x + y}{2}$, show that $\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}$.

- (ii) Hence show that $\sqrt{abc} \leq \frac{a^2 + b^2 + c^2 + a + b + c}{6}$.

(i) $\sqrt{abc} = \sqrt{ab \cdot c} \leq \frac{ab + c}{2}$, and $ab \leq \frac{a^2 + b^2}{2}$, so $\sqrt{abc} \leq \frac{\frac{a^2 + b^2}{2} + c}{2} = \frac{a^2 + b^2 + 2c}{4}$. **(ii)** By symmetry, $\sqrt{abc} \leq \frac{b^2 + c^2 + 2a}{4}$ and $\sqrt{abc} \leq \frac{c^2 + a^2 + 2b}{4}$. Add the three: $3\sqrt{abc} \leq \frac{2(a^2 + b^2 + c^2) + 2(a + b + c)}{4}$. Divide by 3.

In (i) apply AM-GM with $x = ab, y = c$, then use $ab \leq \frac{a^2 + b^2}{2}$ (Lesson 5). In (ii) the left side is the same in every case, which is why adding the three symmetric versions works.

HSC practice • A condition and AM-GM

Let a, b, c be positive real numbers with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that $a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \leq abc$.

AM-GM: $a\sqrt{bc} \leq \frac{a(b+c)}{2}$, $b\sqrt{ca} \leq \frac{b(c+a)}{2}$, $c\sqrt{ab} \leq \frac{c(a+b)}{2}$. **Add:** $\text{LHS} \leq \frac{2ab + 2bc + 2ca}{2} = ab + bc + ca$. **Multiplying the condition by abc gives $bc + ca + ab = abc$. So**
 $\text{LHS} \leq abc$.

Multiply each AM-GM bound by the positive factor (a, b or c). Note that $\sqrt{bc} \leq \frac{b+c}{2}$, so $a\sqrt{bc} \leq \frac{a(b+c)}{2}$. The condition is used by clearing the fractions. (Wording adapted from the paper.)

Example 3

Prove the triangle inequality by showing $(|a| + |b|)^2 - |a + b|^2 \geq 0$.

$(|a| + |b|)^2 - |a + b|^2 = a^2 + 2|a||b| + b^2 - (a^2 + 2ab + b^2) = 2(|ab| - ab) \geq 0$. **Both $|a| + |b|$ and $|a + b|$ are non-negative, so $|a + b| \leq |a| + |b|$.**

Since $ab \leq |ab|$, the bracket $|ab| - ab$ is non-negative. Squares of non-negative numbers compare in the same order as the numbers themselves. The same inequality holds later for complex numbers and for vectors.

Example 4 • Using the triangle inequality

(a) Prove that $|a + b + c| \leq |a| + |b| + |c|$.

$$|a + b + c| \leq |a + b| + |c| \leq |a| + |b| + |c|.$$

Apply the triangle inequality twice: first to $(a + b)$ and c , then to a and b .

(b) Prove that $|a - b| \geq |a| - |b|$. (Hint: write $a = (a - b) + b$.)

$$|a| = |(a - b) + b| \leq |a - b| + |b|, \text{ so } |a - b| \geq |a| - |b|.$$

Apply the triangle inequality to $(a - b)$ and b , then subtract $|b|$ from both sides.

Example 5 • Inequalities in geometry

(a) A rectangle has perimeter P . Prove that its area is at most $\frac{P^2}{16}$. When is there equality?

Let the sides be x, y with $2(x + y) = P$. By AM-GM, $xy \leq \left(\frac{x+y}{2}\right)^2 = \left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$. Equality holds when $x = y$, that is a square.

Use $\sqrt{xy} \leq \frac{x+y}{2}$ and square both sides (both are non-negative). The semi-perimeter $x + y = \frac{P}{2}$ is fixed, so $\frac{x+y}{2} = \frac{P}{4}$.

(b) A right-angled triangle has legs a, b and hypotenuse c . Prove that $a + b \leq c\sqrt{2}$.

$$(a + b)^2 = a^2 + b^2 + 2ab \leq 2(a^2 + b^2) = 2c^2. \text{ Since } a + b > 0, a + b \leq c\sqrt{2}.$$

Use $2ab \leq a^2 + b^2$ and Pythagoras, $a^2 + b^2 = c^2$. Taking the square root is valid because $a + b$ and $c\sqrt{2}$ are both positive.

- (c) Two sides of a triangle are 5 cm and 9 cm. Find the possible lengths of the third side.

$$4 < x < 14.$$

The triangle inequality must hold for each side: $x < 5 + 9 = 14$, $9 < 5 + x$ gives $x > 4$ and $5 < 9 + x$ is always true. So $9 - 5 < x < 9 + 5$.

Extension • AM-GM for four and three numbers

- (i) Using AM-GM for two numbers twice, show that for positive reals $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

$$\frac{a+b+c+d}{4} = \frac{1}{2} \left[\frac{a+b}{2} + \frac{c+d}{2} \right] \geq \frac{1}{2} [\sqrt{ab} + \sqrt{cd}] \geq \sqrt{\sqrt{ab}\sqrt{cd}} = \sqrt[4]{abcd}.$$

The first step uses AM-GM on each pair. The second step uses AM-GM again on the two numbers $\sqrt{ab}$ and $\sqrt{cd}$.

- (ii) By choosing $d = \frac{a+b+c}{3}$ in (i), deduce that $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

Let $m = \frac{a+b+c}{3}$, so $a+b+c = 3m$. Then (i) gives $\frac{3m+m}{4} = m \geq \sqrt[4]{abcm}$. So $m^4 \geq abcm$, hence $m^3 \geq abc$ (as $m > 0$), and $m \geq \sqrt[3]{abc}$.

Putting $d = m$ makes the left side of (i) equal to m itself. Raise both sides to the fourth power and divide by $m > 0$.

Exit ticket

- (a) Prove that $x + \frac{4}{x} \geq 4$ for $x > 0$.

By AM-GM, $x + \frac{4}{x} \geq 2\sqrt{x \cdot \frac{4}{x}} = 2\sqrt{4} = 4$.

The product of the two terms is the constant 4. Equality holds at $x = 2$.

- (b) Two sides of a triangle are 6 and 10. How many integer lengths are possible for the third side?

11.

$10 - 6 < x < 10 + 6$, so $4 < x < 16$. The integers 5 to 15 give 11 values.

Spot the flaw • AM-GM

A student wrote this. Find the flaw, explain why it matters, then write a correct version.

Claim. $x + \frac{1}{x} \geq 2$ for all real $x \neq 0$.

1. By AM-GM, $x + \frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}}$

2. $= 2\sqrt{1} = 2. \square$

The flaw: AM-GM holds only for non-negative numbers, and x may be negative. The claim is false:

$x = -1$ gives $x + \frac{1}{x} = -2 < 2$. **Correct version: for $x > 0$, $x + \frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}} = 2$, with equality at $x = 1$. (For $x < 0$, $x + \frac{1}{x} \leq -2$.)**

For $x < 0$ the product $x \cdot \frac{1}{x} = 1$ is still defined, which hides the problem, but the two terms are negative so AM-GM does not apply. Check the conditions of a theorem before using it, the same habit as dividing only by a positive quantity.

Which method? • Proofs so far

(a) There are no integers m, n with $6m + 9n = 100$.

Contradiction (or a direct divisibility argument). First line: suppose $6m + 9n = 100$ for some integers m, n .

$6m + 9n = 3(2m + 3n)$ is divisible by 3, but 100 is not, which is a contradiction. The words "there are no" point to contradiction.

(b) For integers n , if $n^2 + 2n$ is odd, then n is odd.

Contrapositive. First line: we prove that if n is even, then $n^2 + 2n$ is even.

Let $n = 2k$. Then $n^2 + 2n = 4k^2 + 4k = 2(2k^2 + 2k)$, which is even. "If ... odd, then ... odd" points to the contrapositive.

(c) $\frac{a+b}{2} \geq \frac{2ab}{a+b}$ for $a, b > 0$.

Difference and squares. First line: $\text{LHS} - \text{RHS} = \frac{(a+b)^2 - 4ab}{2(a+b)}$.

This equals $\frac{(a-b)^2}{2(a+b)} \geq 0$, since the numerator is a square and $a+b > 0$.

(d) $n^2 - n$ is even for every integer n .

Direct proof (or cases). First line: $n^2 - n = n(n-1)$.

This is a product of consecutive integers, so one factor is even and the product is even. Alternatively use cases $n = 2k$ and $n = 2k+1$.

(e) $\sqrt{5}$ is irrational.

Contradiction. First line: suppose $\sqrt{5} = \frac{p}{q}$ with p, q integers having no common factor.

Then $p^2 = 5q^2$, so $5 \mid p^2$, which gives $5 \mid p$ (lemma: prove it by cases $p = 5k + r$ with $r = 1, 2, 3, 4$, or by prime factorisation). Writing $p = 5k$ gives $5 \mid q$, contradicting no common factor.

(f) $x + \frac{16}{x} \geq 8$ for $x > 0$.

AM-GM. First line: $x + \frac{16}{x} \geq 2\sqrt{x \cdot \frac{16}{x}}$.

This equals $2\sqrt{16} = 8$, with equality at $x = 4$. A sum of positive terms with a fixed product points to AM-GM.

Practice worksheet

AM-GM

1. Prove that $x + \frac{16}{x} \geq 8$ for $x > 0$, and state when equality holds.

By AM-GM, $x + \frac{16}{x} \geq 2\sqrt{x \cdot \frac{16}{x}} = 2\sqrt{16} = 8$. **Equality holds when** $x = \frac{16}{x}$, **so** $x = 4$.

2. Prove that $(a+1)(b+1)(a+b) \geq 8ab$ for $a, b > 0$.

By AM-GM, $a+1 \geq 2\sqrt{a}$, $b+1 \geq 2\sqrt{b}$ and $a+b \geq 2\sqrt{ab}$. **All are positive, so multiplying:** $(a+1)(b+1)(a+b) \geq 8\sqrt{a}\sqrt{b}\sqrt{ab} = 8ab$.

3. Prove that $|a-b| \leq |a| + |b|$ for real a, b .

$|a-b| = |a+(-b)| \leq |a| + |-b| = |a| + |b|$, **by the triangle inequality.**

4. Prove that $||a| - |b|| \leq |a-b|$. (Hint: use $|a| = |(a-b) + b|$ and $|b| = |(b-a) + a|$.)

$|a| \leq |a-b| + |b|$ **gives** $|a| - |b| \leq |a-b|$. **Also** $|b| \leq |b-a| + |a| = |a-b| + |a|$ **gives** $|b| - |a| \leq |a-b|$. **So** $-|a-b| \leq |a| - |b| \leq |a-b|$, **which means** $||a| - |b|| \leq |a-b|$.

5. Two sides of a triangle are 7 cm and 12 cm. (a) Find the possible lengths of the third side. (b) How many integer lengths are possible? (c) If the triangle is isosceles, find its sides.

(a) $12 - 7 < x < 12 + 7$, **so** $5 < x < 19$. **(b)** **The integers 6 to 18: 13 values.** **(c)** 7, 7, 12 (valid since $14 > 12$) or 7, 12, 12 (valid).

Geometry

6. A rectangle has area 36 cm^2 . Prove that its perimeter is at least 24 cm, and state when it equals 24 cm.

Let the sides be x, y **with** $xy = 36$. **By AM-GM,** $\frac{x+y}{2} \geq \sqrt{xy} = 6$, **so** $x+y \geq 12$ **and the perimeter** $2(x+y) \geq 24$. **Equality holds when** $x = y = 6$ **(a square).**

7. A right-angled triangle has hypotenuse 10 cm. Prove that its perimeter is at most $10 + 10\sqrt{2}$ cm. When is this achieved?

With legs a, b , $(a+b)^2 = a^2 + b^2 + 2ab \leq 2(a^2 + b^2) = 200$, **so** $a+b \leq 10\sqrt{2}$ **(positive).** **Perimeter** $= a+b+10 \leq 10 + 10\sqrt{2}$. **Equality holds when** $a = b = 5\sqrt{2}$ **(isosceles right-angled).**

Adapted HSC-style

8. (a) Prove that $(a + b) \left(\frac{1}{a} + \frac{1}{b} \right) \geq 4$ for $a, b > 0$. [2] (b) Hence show that if $a + b = 1$, then $\frac{1}{a} + \frac{1}{b} \geq 4$. [1] (c) Hence show that if $a, b > 0$ and $a + b = 1$, then $\frac{1}{a^2} + \frac{1}{b^2} \geq 8$. [2]
- (a)** $(a + b) \left(\frac{1}{a} + \frac{1}{b} \right) = 2 + \frac{a}{b} + \frac{b}{a}$, and $\frac{a}{b} + \frac{b}{a} - 2 = \frac{(a - b)^2}{ab} \geq 0$, so the product is at least 4.
- (b)** Put $a + b = 1$ in (a): $1 \times \left(\frac{1}{a} + \frac{1}{b} \right) \geq 4$. **(c)** Using $x^2 + y^2 \geq 2xy$ with $x = \frac{1}{a}$, $y = \frac{1}{b}$: $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{ab} = 2 \left(\frac{1}{a} + \frac{1}{b} \right)$, because $\frac{1}{a} + \frac{1}{b} = \frac{a + b}{ab} = \frac{1}{ab}$. By (b) this is at least $2 \times 4 = 8$.