

Worked solutions

Lesson 5 • Algebraic inequalities I

Warm-up • Inequality instincts

- (a) Solve
- $3 - 2x > 7$
- .

$$x < -2.$$

Subtract 3 to get $-2x > 4$, then divide by -2 . Dividing by a negative number reverses the inequality.

- (b) Complete the square:
- $x^2 - 6x + 11$
- .

$$(x - 3)^2 + 2, \text{ so the expression is always at least 2.}$$

$x^2 - 6x + 9 = (x - 3)^2$, leaving $+2$. A square is never negative, so the minimum value is 2 at $x = 3$.

- (c) True or false: if
- $a > b$
- , then
- $a^2 > b^2$
- .

False. For example $a = 1, b = -2$.

$1 > -2$ but $1^2 = 1 < 4 = (-2)^2$. The statement is true if $a > b > 0$.

- (d) True or false: if
- $a > b$
- , then
- $\frac{1}{a} < \frac{1}{b}$
- .

False. For example $a = 1, b = -1$.

$1 > -1$ but $\frac{1}{1} = 1 > -1 = \frac{1}{-1}$. Both (c) and (d) are true under extra conditions, which are today's properties.

Example 1

Prove that $x^2 + y^2 \geq 2xy$ for all real x and y , using each layout. When does equality hold?

Layout 1: $x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. **Layout 2:** $(x - y)^2 \geq 0 \Rightarrow x^2 - 2xy + y^2 \geq 0 \Rightarrow x^2 + y^2 \geq 2xy$. **Equality holds if and only if** $x = y$.

A square is zero only when what is inside it is zero, so $(x - y)^2 = 0 \Leftrightarrow x = y$. This result is used constantly in Lesson 6 and in HSC questions.

Example 2 • Squares are non-negative

- (a) Prove that
- $a^2 + \frac{1}{a^2} \geq 2$
- for
- $a \neq 0$
- .

$$a^2 + \frac{1}{a^2} - 2 = \left(a - \frac{1}{a}\right)^2 \geq 0.$$

Expand $\left(a - \frac{1}{a}\right)^2 = a^2 - 2 + \frac{1}{a^2}$. The restriction $a \neq 0$ is needed so that $\frac{1}{a^2}$ exists.

(b) Prove that $(x + y)^2 \geq 4xy$ for all real x, y .

$$(x + y)^2 - 4xy = x^2 - 2xy + y^2 = (x - y)^2 \geq 0.$$

Subtract the right-hand side and recognise a perfect square.

(c) Prove that $\frac{a^2 + b^2}{2} \geq \left(\frac{a + b}{2}\right)^2$.

$$\text{LHS} - \text{RHS} = \frac{2a^2 + 2b^2 - a^2 - 2ab - b^2}{4} = \frac{(a - b)^2}{4} \geq 0.$$

In words: the mean of the squares is at least the square of the mean. Use a common denominator of 4 before simplifying.

HSC practice • Squares are non-negative

Let x and y be real numbers with $x^2 + y^2 \neq 0$. By considering $(x - y)^2 \geq 0$, show that $\frac{(x + y)^2}{x^2 + y^2} \leq 2$.

$(x - y)^2 \geq 0 \Rightarrow x^2 + y^2 \geq 2xy \Rightarrow 2(x^2 + y^2) \geq x^2 + 2xy + y^2 = (x + y)^2$. Since $x^2 + y^2 > 0$,
divide both sides by it: $\frac{(x + y)^2}{x^2 + y^2} \leq 2$.

State that $x^2 + y^2 > 0$ before dividing: it is non-negative and not zero. Dividing by a positive number keeps the inequality sign.

Example 3 • Using the properties

(a) Prove that if $a > b > 0$, then $a^2 > b^2$.

$$a^2 - b^2 = (a - b)(a + b), \text{ a product of two positive numbers, so } a^2 > b^2.$$

$a - b > 0$ because $a > b$, and $a + b > 0$ because both are positive.

(b) Prove that if $a, b > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$.

$$\frac{a}{b} + \frac{b}{a} - 2 = \frac{a^2 + b^2 - 2ab}{ab} = \frac{(a - b)^2}{ab} \geq 0, \text{ since } ab > 0.$$

Combine over the common denominator ab . A non-negative numerator over a positive denominator is non-negative.

(c) Prove that $a^3 + b^3 \geq a^2b + ab^2$ for $a, b > 0$.

$$a^3 + b^3 - a^2b - ab^2 = a^2(a - b) - b^2(a - b) = (a - b)(a^2 - b^2) = (a - b)^2(a + b) \geq 0.$$

Group in pairs and factor out $(a - b)$. Then $(a - b)^2 \geq 0$ and $a + b > 0$, so the product is non-negative.

HSC practice • Which property fails?

Which statement is NOT true for all real numbers $a < b$ (in the domain of each function)?

A. $a^3 < b^3$

B. $e^{-a} > e^{-b}$

C. $\ln a < \ln b$

D. $\frac{1}{a} > \frac{1}{b}$

D.

Counterexample: $a = -1, b = 1$ gives $\frac{1}{a} = -1 < 1 = \frac{1}{b}$. The property $\frac{1}{b} > \frac{1}{a}$ holds only when a, b are both positive (or both negative). A, B and C hold because x^3 and $\ln x$ are increasing and e^{-x} is decreasing. (Options reworded from the paper.)

Example 4 • Using a given condition

(a) Prove that $ab \leq \frac{1}{4}$.

$$1 = (a + b)^2 \geq 4ab \text{ (Example 2(b))}, \text{ so } ab \leq \frac{1}{4}.$$

Square the condition $a + b = 1$, then use $(x + y)^2 \geq 4xy$ and divide by 4.

(b) Prove that $a^2 + b^2 \geq \frac{1}{2}$.

$$a^2 + b^2 = (a + b)^2 - 2ab = 1 - 2ab \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

By (a), $2ab \leq \frac{1}{2}$, and subtracting a smaller number from 1 gives a larger result.

(c) Hence prove that $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) \geq 9$.

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = 1 + \frac{a+b}{ab} + \frac{1}{ab} = 1 + \frac{2}{ab} \geq 1 + 2 \times 4 = 9.$$

Expand, then use $a + b = 1$. Since $0 < ab \leq \frac{1}{4}$, we have $\frac{1}{ab} \geq 4$ (taking reciprocals reverses the inequality).

Exam-style • Prove a result, then use it

(i) Prove that $x^2 + y^2 \geq 2xy$ for all real x and y .

$$x^2 + y^2 - 2xy = (x - y)^2 \geq 0.$$

Layout 1 from Example 1.

(ii) Hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$ for all real a, b, c .

Add $a^2 + b^2 \geq 2ab$, $b^2 + c^2 \geq 2bc$ and $c^2 + a^2 \geq 2ca$ to get $2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$.

Halve both sides.

Apply (i) three times, once to each pair of variables, then add the three inequalities.

(iii) Hence, or otherwise, show that if $a + b + c = 3$, then $ab + bc + ca \leq 3$.

$$9 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 3(ab + bc + ca), \text{ so } ab + bc + ca \leq 3.$$

Square the condition, then replace $a^2 + b^2 + c^2$ by the smaller quantity $ab + bc + ca$ using (ii). This is the syllabus idea of using previously obtained inequalities.

Exit ticket

(a) Prove that $x^2 - 4x + 5 > 0$ for all real x .

$$x^2 - 4x + 5 = (x - 2)^2 + 1 \geq 1 > 0.$$

Complete the square. The square is at least 0, so the whole expression is at least 1.

(b) Given $a > b > 0$, prove that $\frac{a+1}{b+1} < \frac{a}{b}$.

$$\frac{a}{b} - \frac{a+1}{b+1} = \frac{a(b+1) - b(a+1)}{b(b+1)} = \frac{a-b}{b(b+1)} > 0.$$

The numerator $a - b > 0$ and the denominator $b(b+1) > 0$, so the difference is positive.

Spot the flaw • Starting from the result

A student wrote this. Find the flaw, explain why it matters, then write a correct version.

Claim. $a^2 + b^2 \geq 2ab$ for all real a and b .

1. $a^2 + b^2 \geq 2ab$

2. $\Rightarrow a^2 - 2ab + b^2 \geq 0$

3. $\Rightarrow (a - b)^2 \geq 0$, which is true. Hence $a^2 + b^2 \geq 2ab$. $\square$

The flaw: the student starts from the statement to be proved and ends at something true. A true ending does not make the start true. **Correct version:** $(a - b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$ (Layout 2), or write $\Leftrightarrow$ between the lines.

From $-1 = 1$, squaring gives $1 = 1$, which is true, yet $-1 = 1$ is false. Here every step happens to be reversible, so the argument can be rescued by writing $\Leftrightarrow$ or by reversing the order. Reversing fails when a step is not reversible, for example squaring both sides, or multiplying by something that may be negative or zero.

Practice worksheet

Squares are non-negative

1. Prove each inequality for all real numbers.

(a) $x^2 - 6x + 10 > 0$

(b) $a^2 + b^2 + 2 \geq 2(a + b)$

(c) $x^2 + y^2 + 1 \geq x + y + xy$

(a) $x^2 - 6x + 10 = (x - 3)^2 + 1 \geq 1 > 0$. (b) $a^2 + b^2 + 2 - 2a - 2b = (a - 1)^2 + (b - 1)^2 \geq 0$. (c) $2(x^2 + y^2 + 1 - x - y - xy) = (x - y)^2 + (x - 1)^2 + (y - 1)^2 \geq 0$, so $x^2 + y^2 + 1 - x - y - xy \geq 0$.

2. Prove that $a^4 + b^4 \geq a^3b + ab^3$ for all real a and b .

$$a^4 + b^4 - a^3b - ab^3 = a^3(a - b) - b^3(a - b) = (a - b)(a^3 - b^3) = (a - b)^2(a^2 + ab + b^2).$$

Now $a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 + \frac{3}{4}b^2 \geq 0$, so the product is non-negative.

Properties of inequalities

3. Give a counterexample to each statement.

(a) If $a > b$ and $c > d$, then $a - c > b - d$.

(b) If $a > b$ and $c > d$, then $\frac{a}{c} > \frac{b}{d}$.

(a) $a = 5, b = 1, c = 4, d = 0$: $a - c = 1$ and $b - d = 1$, so $a - c \not> b - d$. (b) $a = 2, b = 1, c = 3, d = 1$: $a > b$ and $c > d$, but $\frac{a}{c} = \frac{2}{3} < 1 = \frac{b}{d}$.

4. Given $a > b > 0$, prove that $\frac{a}{a+1} > \frac{b}{b+1}$.

$\frac{a}{a+1} - \frac{b}{b+1} = \frac{a(b+1) - b(a+1)}{(a+1)(b+1)} = \frac{a-b}{(a+1)(b+1)} > 0$, since $a - b > 0$ and both brackets are positive.

Using a given condition

5. Let $a, b > 0$ with $a + b = 2$. (a) Prove that $ab \leq 1$. (b) Prove that $a^2 + b^2 \geq 2$. (c) Prove that $\frac{1}{a} + \frac{1}{b} \geq 2$.

(a) $(a + b)^2 - 4ab = (a - b)^2 \geq 0$, so $4 \geq 4ab$ and $ab \leq 1$. (b) $a^2 + b^2 = (a + b)^2 - 2ab = 4 - 2ab \geq 4 - 2 = 2$. (c) $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{2}{ab} \geq 2$, because $0 < ab \leq 1$.

6. For $a, b, c > 0$, prove that $(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9$.

Expanding, the product equals $3 + \left(\frac{a}{b} + \frac{b}{a} \right) + \left(\frac{b}{c} + \frac{c}{b} \right) + \left(\frac{a}{c} + \frac{c}{a} \right)$. Each bracket is at least 2 (as $\frac{x}{y} + \frac{y}{x} \geq 2$ for positive x, y), so the product is at least $3 + 6 = 9$.

Spot the flaw

7. A student "proves" that $x^2 + 1 \geq 2x$ as follows: " $x^2 + 1 \geq 2x \Rightarrow x^2 - 2x + 1 \geq 0 \Rightarrow (x - 1)^2 \geq 0$, which is true, so the claim holds." Explain the flaw and write a correct proof.

The student starts from the claim and ends at a true statement, which proves nothing by itself.

Correct proof: $x^2 + 1 - 2x = (x - 1)^2 \geq 0$, so $x^2 + 1 \geq 2x$.

Adapted HSC-style

8. (a) Prove that $x^2 + y^2 \geq 2xy$ for all real x, y . [1] (b) Hence show that $(a + b)^2 \leq 2(a^2 + b^2)$ for all real a, b . [1] (c) Hence show that if $a + b + c = 1$, then $a^2 + b^2 + c^2 \geq \frac{1}{3}$. [2]

(a) $x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. (b) $2(a^2 + b^2) - (a + b)^2 = a^2 - 2ab + b^2 = (a - b)^2 \geq 0$.

(c) Add $a^2 + b^2 \geq 2ab, b^2 + c^2 \geq 2bc, c^2 + a^2 \geq 2ca$ to get $a^2 + b^2 + c^2 \geq ab + bc + ca$. Then $1 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \leq 3(a^2 + b^2 + c^2)$, so $a^2 + b^2 + c^2 \geq \frac{1}{3}$.