

Worked solutions

Lesson 1 • Logic and the language of proof

Warm-up • True or false?

- (a) All prime numbers are odd.

False.

2 is prime and even, so 2 is a counterexample. One counterexample is enough to disprove a “for all” claim.

- (b) $n^2 + n + 41$ is prime for every $n \in \mathbb{N}$.

False, although it works for $n = 0$ to 39.

For $n = 40$: $1600 + 40 + 41 = 1681 = 41^2$. Many examples never prove a statement about infinitely many cases, but one counterexample disproves it.

- (c) The sum of two odd numbers is even.

True.

Checking examples such as $3 + 5 = 8$ is not a proof. We prove this in Lesson 3.

- (d) If $x^2 = 9$, then $x = 3$.

False.

$x = -3$ satisfies $x^2 = 9$ but not $x = 3$, so it is a counterexample.

Example 1 • Which are statements? If so, true or false?

- (a) There are kangaroos in Australia.

A statement, and it is true.

It is a sentence that is either true or false.

- (b) Close the door.

Not a statement.

It is a command, so it is neither true nor false.

- (c) $x + 1 = 5$

Not a statement until x is known.

Its logical value depends on x . It becomes a statement with a quantifier, for example $\exists x \in \mathbb{R}$ such that $x + 1 = 5$, which is true (Lesson 2).

(d) 16 is an odd number.

A statement, and it is always false.

A statement can be false. What matters is that it has a definite logical value.

(e) History is more interesting than Science.

Not a statement.

It is an opinion, which has no logical value.

Example 2 • Exhaustion and counterexample

(a) Prove by exhaustion: all odd integers between 16 and 20 are prime.

The odd integers between 16 and 20 are 17 and 19. Both are prime, so the statement is proved.

There are only two cases, so we can check them all. That is what makes the proof valid.

(b) Disprove: all primes are odd.

2 is prime and even, so the statement is false.

A single counterexample is enough to disprove a “for all” statement.

HSC practice • A counterexample

Consider the claim: “If n is prime, then $2^n - 1$ is prime.” Which value of n is a counterexample?

A $n = 2$

B $n = 3$

C $n = 7$

D $n = 11$

D

$2^{11} - 1 = 2047 = 23 \times 89$, and 11 is prime. For A, B and C: $2^2 - 1 = 3$, $2^3 - 1 = 7$ and $2^7 - 1 = 127$ are all prime, so they agree with the claim and are not counterexamples. A counterexample must make the “if” part true and the “then” part false.

Example 3 • Negation

1. P : It is raining.

It is not raining.

The simplest case: $\neg P$ is “not P ”.

2. Q : All dogs are brown.

Not all dogs are brown: there is a dog that is not brown.

Do not write “all dogs are not brown”. That says no dog is brown, which is too strong.

3. R : The bottle of water is full.

The bottle of water is not full.

Not “empty”. A half-full bottle is neither full nor empty.

4. S : Some students ride to school.

No students ride to school.

The opposite of “some” is “none”.

5. $T: x > 4$

$x \leq 4$

Not $x < 4$. The negation must include the boundary case $x = 4$.

6. $U: -3 \leq x \leq 8$

$x < -3$ or $x > 8$

The “and” hidden in $-3 \leq x \leq 8$ (that is, $x \geq -3$ and $x \leq 8$) becomes “or”. This previews the negation rules later in the lesson.

Example 4 • P : 6 is even, Q : 6 is prime

(a) Is $P \wedge Q$ true?

No, $P \wedge Q$ is false.

P is true and Q is false. Both must be true for “and”.

(b) Is $P \vee Q$ true?

Yes, $P \vee Q$ is true.

At least one must be true, and 6 is even.

Implication

When has Bob lied? Complete the truth table for $P \Rightarrow Q$, $\neg P$ and $\neg P \vee Q$. Then complete: $P \Rightarrow Q$ is false only when ...

$P \Rightarrow Q$ is false only when P is true and Q is false.

Bob only lies when Jane pays and he does not go shopping (row 2). If Jane never pays, Bob cannot have broken his promise, so $P \Rightarrow Q$ is true. This is called being vacuously true. The columns for $P \Rightarrow Q$ and $\neg P \vee Q$ are identical, so $P \Rightarrow Q \simeq \neg P \vee Q$.

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

Example 5 • Testing an implication

P : x is a multiple of 6. Q : x is divisible by 9. Complete a truth table for $P \Rightarrow Q$, giving a value of x for each row. Is the implication always true?

No. $x = 24$ is a counterexample, so $P \Rightarrow Q$ is not always true.

One value of x for each row:

P	Q	Example of x	$P \Rightarrow Q$
1	1	18	1
1	0	24	0
0	1	27	1
0	0	25	1

Rows 3 and 4 are true because P is false (vacuously true). Row 2 is the only row where the implication fails: 24 is a multiple of 6 but not divisible by 9.

Example 6 • Negating compound statements

(a) $x > 2$ and x is even.

$x \leq 2$ or x is odd.

Negating “and” gives “or”, and each part is negated: $\neg(P \wedge Q) \simeq \neg P \vee \neg Q$. (For an integer x , “not even” means odd.)

(b) n is odd or n is a multiple of 3.

n is even and n is not a multiple of 3.

Negating “or” gives “and”: $\neg(P \vee Q) \simeq \neg P \wedge \neg Q$. Both parts have to fail.

(c) If n is even, then n^2 is even.

n is even and n^2 is odd.

$\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$. Keep the hypothesis and deny the conclusion. Do not write “if n is even then n^2 is odd”, which is another implication.

(d) If I live in Tasmania, then I live in Australia.

I live in Tasmania and I do not live in Australia.

Again $P \wedge \neg Q$. (The original is true, so this negation is false for everyone.)

Exit ticket

1. Which is the negation of “If n is prime, then n is odd”?

A n is not prime and n is even.

B n is prime and n is even.

C If n is odd, then n is prime.

D n is not prime or n is odd.

B

$\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$: “ n is prime and n is not odd”, and not odd means even. A negates both parts. C is the converse. D is equivalent to the original statement ($\neg P \vee Q$).

2. Which is logically equivalent to $P \Rightarrow Q$?

A $Q \Rightarrow P$

B $P \wedge \neg Q$

C $\neg P \vee Q$

D $\neg P \wedge Q$

C

$P \Rightarrow Q \simeq \neg P \vee Q$, as the truth table shows. B is the negation of $P \Rightarrow Q$. A is the converse. D is false when P and Q are both true, but $P \Rightarrow Q$ is true then.

Practice worksheet

Statements

1. Decide which of these are statements. For each statement, say whether it is true or false.

(a) 7 is a prime number.

(b) How old are you?

(c) $2x + 3 = 11$

(d) Every square is a rectangle.

(a) A statement, true. (b) Not a statement: it is a question. (c) Not a statement until x is known. (d) A statement, true.

2. Let P : " n is divisible by 4" and Q : " n is even". For $n = 6$, state whether each is true or false: (a) $P \wedge Q$ (b) $P \vee Q$ (c) $\neg P$ (d) $P \Rightarrow Q$.

P is false and Q is true when $n = 6$. (a) False, since P is false. (b) True, since Q is true. (c) True. (d) True, since P is false (an implication with a false hypothesis is true).

Negation

3. Write the negation of each statement.

(a) All triangles have a right angle.

(b) $x \geq 3$

(c) n is prime and n is odd.

(d) a is positive or b is negative.

(e) If $x > 0$, then $x^2 > 0$.

(a) Not all triangles have a right angle: there is a triangle with no right angle. (b) $x < 3$. (c) n is not prime or n is even. (d) $a \leq 0$ and $b \geq 0$. (e) $x > 0$ and $x^2 \leq 0$.

4. Complete a truth table (using 1 and 0) to show that $\neg(P \wedge Q) \simeq \neg P \vee \neg Q$.

The columns for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are identical, so the statements are logically equivalent.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

5. Use $P \Rightarrow Q \simeq \neg P \vee Q$ to show that $\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$.

$\neg(P \Rightarrow Q) \simeq \neg(\neg P \vee Q) \simeq \neg(\neg P) \wedge \neg Q \simeq P \wedge \neg Q$.

Implication, exhaustion and counterexamples

6. Let P : x is a multiple of 4 and Q : x is a multiple of 8. Find a value of x for each row of the truth table for $P \Rightarrow Q$ (rows $P, Q = 1,1; 1,0; 0,1; 0,0$). Is $P \Rightarrow Q$ always true?

For example: (1,1) $x = 16$, true. (1,0) $x = 12$, false. (0,1) impossible, since every multiple of 8 is a multiple of 4. (0,0) $x = 5$, true. So $P \Rightarrow Q$ is not always true: $x = 12$ is a counterexample.

7. (a) Prove by exhaustion: all prime numbers between 20 and 30 are odd. (b) Use a counterexample to show that the statement "For all real x , $x^2 > x$ " is false.

(a) The primes between 20 and 30 are 23 and 29. Both are odd, so the statement is proved. (b) Take $x = 1$: $x^2 = 1$, which is not greater than $x = 1$. (Also $x = 0$ or $x = \frac{1}{2}$ work.) One counterexample disproves a "for all" statement.

Adapted HSC-style

8. **Adapted HSC-style.** Consider: "If n is a multiple of 4 and a multiple of 6, then n is a multiple of 24." (a) Write the negation. (b) Is the original statement true? Justify your answer.

(a) n is a multiple of 4 and a multiple of 6, and n is not a multiple of 24. (b) False. Take $n = 12$: it is a multiple of 4 and of 6, but not of 24. So the negation is true.