

Worked solutions

Lesson 9 • Lines and circles in the Argand diagram

Retrieval and coordinate geometry

- (a) Write the equation of the circle with centre $(1, -2)$ and radius 3.

$$(x - 1)^2 + (y + 2)^2 = 9$$

A circle with centre (a, b) and radius r has equation $(x - a)^2 + (y - b)^2 = r^2$.

- (b) Find the perpendicular bisector of the interval from $(0, 0)$ to $(4, 2)$.

$$y = -2x + 5$$

The midpoint is $(2, 1)$ and the interval has gradient $\frac{1}{2}$, so the bisector has gradient -2 : $y - 1 = -2(x - 2)$.

- (c) Find $|3 - 4i|$.

$$5$$

$$\sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

- (d) Write $z + 2 - i$ in the form $z - (\quad)$.

$$z - (-2 + i)$$

$z + 2 - i = z - (-2 + i)$. Parts (a), (b) and (d) are the three skills this lesson needs.

Example 1 • Sketching circles

- (a) $|z| = 3$

$$x^2 + y^2 = 9: \text{centre } O, \text{ radius } 3.$$

$|z - 0| = 3$: the distance from O is always 3.

- (b) $|z - (1 - 2i)| = 2$

$$(x - 1)^2 + (y + 2)^2 = 4: \text{centre } (1, -2), \text{ radius } 2.$$

The centre is 1 unit from the imaginary axis and the radius is 2, so the circle cuts the imaginary axis.

Example 2 • Regions bounded by circles

- (a) $|z + 3 - i| \leq 2$

The closed disc with centre $(-3, 1)$ and radius 2, solid boundary, shaded inside.

Rewrite as $|z - (-3 + i)| \leq 2$. The inequality is " $\leq$ ", so the circle itself is included.

(b) $|z - 2i| > 1$

Everything outside the circle with centre $(0, 2)$ and radius 1, with a dashed boundary.

$|z - 2i| > 1$ means the distance from $2i$ is more than 1. The inequality is strict, so the circle is not included.

Example 3

Sketch $|z - (1 + i)| = |z - (2 - i)|$ and find its Cartesian equation using two methods.

$$2x - 4y - 3 = 0$$

Method A. Midpoint $(\frac{3}{2}, 0)$. Gradient of Z_1Z_2 is $\frac{-1-1}{2-1} = -2$, so the bisector has gradient $\frac{1}{2}$: $y = \frac{1}{2}(x - \frac{3}{2})$, giving $2x - 4y - 3 = 0$.

Method B. $(x - 1)^2 + (y - 1)^2 = (x - 2)^2 + (y + 1)^2$ gives $-2x - 2y + 2 = -4x + 2y + 5$, so $2x - 4y - 3 = 0$.

Example 4 • A circle and a line

(i) On the same diagram sketch $|z - (3 + 2i)| = 2$ and $|z + 3| = |z - 5|$. (ii) Find the complex number(s) satisfying both equations.

(i) A circle with centre $(3, 2)$ and radius 2, and the vertical line $x = 1$ (the perpendicular bisector of -3 and 5). (ii) The line is tangent to the circle: $z = 1 + 2i$.

The leftmost point of the circle is $x = 3 - 2 = 1$, so the line $x = 1$ touches the circle at exactly one point, $(1, 2)$.

Example 5 • Touching circles

For what values of $k > 0$ does the circle $|z - 2i| = k$ meet $|z - (3 + 2i)| = 2$ at exactly one point?

$$k = 1 \text{ or } k = 5.$$

The centres $(0, 2)$ and $(3, 2)$ are 3 apart. External tangency: $k + 2 = 3$, so $k = 1$. Internal tangency (the smaller circle inside the larger): $k - 2 = 3$, so $k = 5$.

Example 6

Find the locus of w if $w = \frac{z - 1}{z}$ and $|z| = 2$.

The circle $|w - 1| = \frac{1}{2}$: centre 1, radius $\frac{1}{2}$.

Make z the subject: $wz = z - 1$, so $z(w - 1) = -1$ and $z = \frac{-1}{w - 1}$. Then $|z| = 2$ gives

$\frac{1}{|w - 1|} = 2$, so $|w - 1| = \frac{1}{2}$. In Cartesian form, $(u - 1)^2 + v^2 = \frac{1}{4}$ for $w = u + iv$. **Strategy:** if w is given in terms of z , make z the subject, then apply the condition on z .

Example 7

Find the locus of z if $w = \frac{z-2}{z}$ is purely imaginary.

The circle $(x-1)^2 + y^2 = 1$, with the points $z = 0$ and $z = 2$ excluded.

$w = \frac{(x-2+iy)(x-iy)}{x^2+y^2} = \frac{x^2+y^2-2x+2iy}{x^2+y^2}$. Purely imaginary means $\operatorname{Re}(w) = 0$ (and $w \neq 0$), so $x^2+y^2-2x = 0$, that is $(x-1)^2 + y^2 = 1$. Exclude $z = 0$ (where w is undefined) and $z = 2$ (where $w = 0$, which is real). Mark both with open circles.

Example 8

Describe the locus given by $|z+2| = 2|z-2+i|$.

A circle with centre $\left(\frac{10}{3}, -\frac{4}{3}\right)$ and radius $\frac{2\sqrt{17}}{3}$.

$(x+2)^2 + y^2 = 4[(x-2)^2 + (y+1)^2]$ gives $3x^2 + 3y^2 - 20x + 8y + 16 = 0$, so $\left(x - \frac{10}{3}\right)^2 + \left(y + \frac{4}{3}\right)^2 = \frac{68}{9}$. When the ratio of distances is not 1, the locus is a circle, not a line.

Spot the error • Circles and bisectors

(a) $|z+3-i| = 2$ is a circle with centre $(3, -1)$.

The centre is $(-3, 1)$.

Write $|z+3-i| = |z - (-3+i)|$, so the centre is $-3+i$, that is $(-3, 1)$. The sign flips when you read off the centre.

(b) $|z-2i| < 1$ is drawn with a solid boundary.

The boundary should be dashed.

The inequality is strict, so the points on the circle are not in the region.

(c) $|2z-4i| = 6$ is a circle with radius 6.

The radius is 3 and the centre is $2i$.

$|2z-4i| = 2|z-2i| = 6$, so $|z-2i| = 3$. Always factor out the coefficient of z first.

(d) $|z-1| = |z-i|$ is the circle through 1 and i .

It is a line, not a circle: the line $y = x$.

Equal distances from 1 and i give the perpendicular bisector of the interval joining them, which is $y = x$.

Exit ticket

1. Sketch $|z-1-i| = |z+1+i|$ and give its equation.

The line $y = -x$.

This is the perpendicular bisector of $(1, 1)$ and $(-1, -1)$. The midpoint is O and the interval has gradient 1, so the bisector passes through O with gradient -1 .

2. Write the Cartesian equation of $|2z - 4i| = 6$ and sketch it.

$$x^2 + (y - 2)^2 = 9.$$

Divide by 2: $|z - 2i| = 3$, a circle with centre $(0, 2)$ and radius 3.

Practice worksheet

Circles and regions

1. Sketch each locus or region and write its Cartesian equation.

(a) $|z - 2 + i| = 3$

(b) $|z + 1| \leq 1$

(c) $|z - 2i| > 2$

(a) Circle, centre $(2, -1)$, radius 3: $(x - 2)^2 + (y + 1)^2 = 9$. (b) Closed disc, centre $(-1, 0)$, radius 1: $(x + 1)^2 + y^2 \leq 1$. (c) Outside the dashed circle with centre $(0, 2)$, radius 2: $x^2 + (y - 2)^2 > 4$.

Perpendicular bisectors

2. Find the Cartesian equation of $|z - 3| = |z + i|$.

$(x - 3)^2 + y^2 = x^2 + (y + 1)^2$ gives $-6x + 9 = 2y + 1$, so $3x + y - 4 = 0$.

3. The circle $|z - 1 - i| = k$ touches the locus $|z - 1| = |z - 3|$. Find k and the point of contact.

The locus is the line $x = 2$. The centre $(1, 1)$ is 1 unit from it, so $k = 1$ and the point of contact is $2 + i$.

Change of variable

4. If $|z - 1| = 1$ and $w = \frac{1}{z}$, find the locus of w .

$z = \frac{1}{w}$, so $|\frac{1}{w} - 1| = 1$, giving $|1 - w| = |w|$. The locus is the perpendicular bisector of 0 and 1: the line $\operatorname{Re}(w) = \frac{1}{2}$.

5. Find the locus of z if $\frac{z - 1}{z + 1}$ is purely imaginary.

$\frac{(x - 1 + iy)(x + 1 - iy)}{(x + 1)^2 + y^2}$ has real part $\frac{x^2 + y^2 - 1}{(x + 1)^2 + y^2}$. Setting this to 0 gives the circle $x^2 + y^2 = 1$ with $z = \pm 1$ excluded.

6. If $w = 2z + 1$ and $|z| = 3$, find the locus of w .

$z = \frac{w-1}{2}$, so $|w - 1| = 6$: a circle with centre 1 and radius 6.

HSC-style

7. (a) On one Argand diagram sketch $|z - (1 + i)| = 2$ and $|z - 2| = |z + 2i|$. (b) Find the complex numbers that satisfy both equations.

(a) A circle with centre $(1, 1)$ and radius 2, and the line $y = -x$ (the perpendicular bisector of 2 and $-2i$, through the midpoint $(1, -1)$ with gradient -1). (b) Substituting $y = -x$: $(x - 1)^2 + (x + 1)^2 = 4$, so $x^2 = 1$ and $z = 1 - i$ or $z = -1 + i$.