

Worked solutions

Lesson 8 • Rotation, dilation and geometry

Example 1

Let $z = 2 + i$. Plot z , iz , $i^2z = -z$ and $i^3z = -iz$. Describe what multiplying by i does.

$iz = -1 + 2i$, $-z = -2 - i$, $-iz = 1 - 2i$. **Multiplying by i rotates a point 90° anticlockwise about O .**

Each multiplication by $i = \text{cis } \frac{\pi}{2}$ keeps the modulus and adds $\frac{\pi}{2}$ to the argument. The four points form a square centred at O .

Warm-up • Retrieval from Lessons 5 to 7

- (a) Find iz for $z = 3 - 2i$.

$$iz = 2 + 3i$$

$$i(3 - 2i) = 3i - 2i^2 = 2 + 3i, \text{ using } i^2 = -1.$$

- (b) Write $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$ as $x + iy$.

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Use the exact values $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

- (c) Z is $1 + i$ and W is $3 + 4i$. Which complex number does $\overrightarrow{WZ}$ represent?

$$z - w = -2 - 3i$$

Head minus tail: the vector from W to Z represents $z - w$.

- (d) Find the length WZ in part (c).

$$WZ = \sqrt{13}$$

$$WZ = |z - w| = |-2 - 3i| = \sqrt{4 + 9} = \sqrt{13}.$$

Example 2

Let $z = 2 + 2i$. Find the complex number obtained by rotating z anticlockwise by $\frac{\pi}{3}$ about O and doubling its distance from O .

$$(2 - 2\sqrt{3}) + (2 + 2\sqrt{3})i$$

Multiply by $2 \text{cis } \frac{\pi}{3} = 1 + \sqrt{3}i$: $(2 + 2i)(1 + \sqrt{3}i) = 2 + 2\sqrt{3}i + 2i - 2\sqrt{3}$. Check: $|z| = 2\sqrt{2}$ becomes $4\sqrt{2}$, and $\arg z = \frac{\pi}{4}$ becomes $\frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$.

Example 3 • A square

- (a) Find C and B .

$$C = -1 + 2i \text{ and } B = 1 + 3i.$$

Rotating $\overrightarrow{OA}$ by 90° anticlockwise gives $\overrightarrow{OC}$: $c = i(2 + i) = -1 + 2i$. Then $b = a + c = (2 + i) + (-1 + 2i) = 1 + 3i$ since $OABC$ is a parallelogram.

- (b) The square is rotated 45° anticlockwise about O . Find the new position of B .

$$-\sqrt{2} + 2\sqrt{2}i$$

$$b' = b \operatorname{cis} \frac{\pi}{4} = (1 + 3i) \cdot \frac{1+i}{\sqrt{2}} = \frac{-2+4i}{\sqrt{2}} = -\sqrt{2} + 2\sqrt{2}i.$$

Example 4

In triangle ABC , $AB = BC$ and $\angle ABC = 90^\circ$. B is $5 + 3i$ and C is $9 + 6i$, and A lies above the line BC . Find the complex number represented by A .

$$A = 2 + 7i$$

$\overrightarrow{BC} = c - b = 4 + 3i$. Since $BA = BC$ and $\angle ABC = 90^\circ$ with A above BC , $\overrightarrow{BA}$ is $\overrightarrow{BC}$ rotated 90° anticlockwise: $i(4 + 3i) = -3 + 4i$. So $a = b + (-3 + 4i) = 2 + 7i$. (If A were below BC , multiply by $-i$ to get $a = 8 - i$.)

Example 5 • Equilateral triangle

- (i) Explain why $\triangle OAB$ is equilateral. (ii) Show that $z_1^2 - z_1 z_2 + z_2^2 = 0$.

(i) $|z_2| = |z_1|$, so $OA = OB$, and $\angle AOB = \frac{\pi}{3}$. An isosceles triangle with apex angle 60° is equilateral. (ii) Let $\omega = \operatorname{cis} \frac{\pi}{3}$. Then $\omega^3 = \operatorname{cis} \pi = -1$, so $(\omega + 1)(\omega^2 - \omega + 1) = 0$. Since $\omega \neq -1$, $\omega^2 - \omega + 1 = 0$, and $z_1^2 - z_1 z_2 + z_2^2 = z_1^2(1 - \omega + \omega^2) = 0$.

Alternatively, $\omega^2 = \operatorname{cis} \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega - 1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ are equal, so $\omega^2 = \omega - 1$. (The original paper writes z_2 in exponential form.)

Example 6 • Perpendicular vectors

- (i) Prove that $\left| \frac{z-w}{z+w} \right| = 1$. (ii) Interpret this result geometrically.

(i) $\frac{z}{w} = -ri$ for some $r > 0$. Dividing top and bottom by w : $\frac{z-w}{z+w} = \frac{z/w - 1}{z/w + 1} = \frac{-1 - ri}{1 - ri}$. Both numerator and denominator have modulus $\sqrt{1 + r^2}$, so the modulus is 1. (ii) $OZ \perp OW$, so the parallelogram on OZ and OW is a rectangle and its diagonals $|z+w|$ and $|z-w|$ are equal. $\arg \frac{z}{w} = -\frac{\pi}{2}$ means $\frac{z}{w}$ is a negative imaginary number. Rotating w by -90° gives a vector in the direction of z , so $OZ \perp OW$ (compare the exit ticket in Lesson 7).

Example 7 • Proving a quadrilateral is a square

- (a) If $z_1 - z_2 = z_4 - z_3$, show that $ABCD$ is a parallelogram.

$z_1 - z_2$ represents $\overrightarrow{BA}$ and $z_4 - z_3$ represents $\overrightarrow{CD}$. Equal vectors mean $BA \parallel CD$ and $BA = CD$, so $ABCD$ is a parallelogram.

Name the vector each complex number represents before interpreting it. One pair of sides equal and parallel is enough for a parallelogram.

- (b) If also $z_3 - z_2 = i(z_1 - z_2)$, show that $ABCD$ is a square.

$z_3 - z_2 = \overrightarrow{BC}$ is $\overrightarrow{BA}$ rotated through 90° , so $BC = BA$ and $\angle ABC = 90^\circ$. A parallelogram with a right angle and two equal adjacent sides is a square.

Multiplying by i keeps the length and rotates by 90° .

Exit ticket

1. O , 2 and z form an equilateral triangle with z in the first quadrant. Find z .

$$z = 1 + \sqrt{3}i$$

$z = 2 \operatorname{cis} \frac{\pi}{3}$: rotate the vector 2 about O through 60° anticlockwise (the third vertex is at the same distance, 2, from O).

2. $ABCD$ is a square labelled anticlockwise, with A at $1 + i$ and B at $4 + 2i$. Find C and D .

$$C = 3 + 5i \text{ and } D = 4i.$$

$$\overrightarrow{AB} = 3 + i. \overrightarrow{BC} = i(3 + i) = -1 + 3i, \text{ so } C = (4 + 2i) + (-1 + 3i) = 3 + 5i. D = A + \overrightarrow{BC} = (1 + i) + (-1 + 3i) = 4i.$$

Practice worksheet

Rotation and dilation

1. Find the image of $z = 3 - i$ when it is (a) rotated 90° anticlockwise about O , (b) rotated 90° clockwise about O , (c) rotated 60° anticlockwise about O and tripled in distance from O .

$$\text{(a) } i(3 - i) = 1 + 3i. \text{ (b) } -i(3 - i) = -1 - 3i. \text{ (c) } 3 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) (3 - i) = \frac{3}{2}(1 + \sqrt{3}i)(3 - i) = \frac{3}{2}((3 + \sqrt{3}) + (3\sqrt{3} - 1)i).$$

2. The point P represents $4 + i$. Find the point Q obtained by rotating P through $\frac{\pi}{2}$ anticlockwise about the point A representing $1 + 2i$.

$$q - a = i(p - a) = i(3 - i) = 1 + 3i, \text{ so } q = 2 + 5i.$$

Vertices of shapes

3. $OABC$ is a square, labelled anticlockwise, with B at $4 + 6i$. Find A and C .

$$\text{The centre is } 2 + 3i \text{ and } \overrightarrow{OA} \text{ is } \overrightarrow{OB} \text{ rotated } -45^\circ \text{ and scaled by } \frac{1}{\sqrt{2}}: a = (4 + 6i) \cdot \frac{1-i}{2} = 5 + i.$$

$$\text{Then } c = ia = -1 + 5i. \text{ Check: } a + c = 4 + 6i.$$

4. A and B represent $1 + i$ and $5 + 3i$. Find the third vertex C of the equilateral triangle ABC lying above AB (anticlockwise order A, B, C).

$$\overrightarrow{AB} = 4 + 2i. \overrightarrow{AC} = \overrightarrow{AB} \operatorname{cis} \frac{\pi}{3} = (4 + 2i) \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = (2 - \sqrt{3}) + (1 + 2\sqrt{3})i. \text{ So } C = (3 - \sqrt{3}) + (2 + 2\sqrt{3})i.$$

Proofs

5. Show that if $z_2 = iz_1$ then z_1, z_2 and the origin form an isosceles right-angled triangle.

$$|z_2| = |i||z_1| = |z_1|, \text{ so } OZ_1 = OZ_2. \arg z_2 = \arg z_1 + \frac{\pi}{2}, \text{ so } \angle Z_1 O Z_2 = 90^\circ.$$

6. A, B, C represent z_1, z_2, z_3 . If $z_3 - z_1 = i(z_2 - z_1)$, show that $\triangle ABC$ is right-angled and isosceles, and state the right angle.

$$\overrightarrow{AC} = i \overrightarrow{AB}, \text{ so } AC = AB \text{ and } \angle BAC = 90^\circ. \text{ The right angle is at } A.$$

HSC-style

7. Adapted HSC-style. The complex numbers $0, z$ and w are the vertices of an equilateral triangle, labelled anticlockwise. (a) Show that $w = z \operatorname{cis} \frac{\pi}{3}$. (b) Show that $z^2 + w^2 = zw$. (c) Hence show that $\frac{z}{w} + \frac{w}{z} = 1$.

$$\text{(a) } OW = OZ \text{ and } \angle ZOW = \frac{\pi}{3} \text{ anticlockwise, so } w = z \operatorname{cis} \frac{\pi}{3}. \text{ (b) With } \omega = \operatorname{cis} \frac{\pi}{3}, \omega^2 - \omega + 1 = 0 \text{ (since } \omega^3 = -1, \omega \neq -1), \text{ so } z^2 + w^2 - zw = z^2(1 + \omega^2 - \omega) = 0. \text{ (c) Divide } z^2 + w^2 = zw \text{ by } zw \neq 0.$$