

Worked solutions

Lesson 7 • Complex numbers as vectors

Warm-up • Retrieval from Lessons 4 to 6

- (a) Find $|z|$ if $z = (3 + 4i)(1 - i)$.

$$|z| = 5\sqrt{2}$$

The modulus of a product is the product of the moduli: $|3 + 4i||1 - i| = 5 \cdot \sqrt{2}$.

- (b) Find $\text{Arg}(z_1 z_2)$ if $\arg z_1 = \frac{3\pi}{4}$ and $\arg z_2 = \frac{\pi}{2}$.

$$\text{Arg}(z_1 z_2) = -\frac{3\pi}{4}$$

Arguments add: $\frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$. This is outside $(-\pi, \pi]$, so subtract 2π to get the principal argument $-\frac{3\pi}{4}$.

- (c) For $\underline{a} = \underline{3i} + \underline{2j}$ and $\underline{b} = \underline{i} - \underline{2j}$, find $\underline{a} + \underline{b}$ and $\underline{a} - \underline{b}$.

$$\underline{a} + \underline{b} = \underline{4i} \text{ and } \underline{a} - \underline{b} = \underline{2i} + \underline{4j}.$$

Add or subtract components. These are the same numbers as $u + v = 4$ and $u - v = 2 + 4i$ in Example 3, which you met in the Extension 1 vectors work.

- (d) Plot $z = -1 + 2i$ and iz . Through what angle has z turned about O ?

$iz = -2 - i$, and z has turned through 90° anticlockwise.

$iz = i(-1 + 2i) = -i - 2$. Multiplying by $i = \text{cis } \frac{\pi}{2}$ adds $\frac{\pi}{2}$ to the argument.

Example 1

Plot $z = 3 + 4i$ and $\bar{z}$ as vectors from O . Draw the vector from Z to $\bar{Z}$. Which complex number does it represent?

$$\bar{z} - z = (3 - 4i) - (3 + 4i) = -8i: \text{a vector pointing straight down, of length 8.}$$

The vector from Z to $\bar{Z}$ is “head minus tail”, $\bar{z} - z$. The same vector $-8i$ can be drawn starting anywhere: only length and direction matter.

Example 2 • Adding on the Argand diagram

- (a) Show $(1 + 2i) + (3 + i)$ on the Argand diagram.

$(1 + 2i) + (3 + i) = 4 + 3i$. With $\overrightarrow{OA} = 1 + 2i$ and $\overrightarrow{OB} = 3 + i$, the diagonal $\overrightarrow{OC} = 4 + 3i$ of parallelogram $OACB$.

Add real parts and imaginary parts. Geometrically, slide the vector OB so its tail is at A (tip to tail): it ends at C .

- (b) If $z = 1 + 3i$ and $w = 4 + i$, plot z , w , $-z$ and $z + w$.

$z + w = 5 + 4i$ and $-z = -1 - 3i$.

$-z$ is z rotated by 180° about O (same length, opposite direction). $z + w$ is the diagonal of the parallelogram on z and w .

Example 3 • Subtraction, tip to tail

1. $u + v$

$u + v = 4$ (a horizontal vector of length 4).

Place the tail of v at the tip of u , or take the long diagonal of the parallelogram on u and v .

2. $u - v$

$u - v = u + (-v) = 2 + 4i$: the vector from V to U .

Head minus tail: the vector pointing from the tip of v to the tip of u . It is the other diagonal of the parallelogram.

3. $v - u$

$v - u = -2 - 4i$: the vector from U to V .

This is the opposite of the vector in part 2: same length, reversed direction.

Example 4

$OABC$ is a parallelogram with A at $5 + i$ and C at $2 + 3i$. Find the complex numbers represented by

(a) B , (b) $\overrightarrow{AC}$, (c) $\overrightarrow{CA}$, and (d) the lengths of both diagonals.

(a) $b = 7 + 4i$. (b) $\overrightarrow{AC} = -3 + 2i$. (c) $\overrightarrow{CA} = 3 - 2i$. (d) $OB = \sqrt{65}$ and $AC = \sqrt{13}$.

(a) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC} = (5 + i) + (2 + 3i)$. (b) head minus tail: $c - a = (2 + 3i) - (5 + i)$. (c) the opposite vector, $a - c$. (d) $|7 + 4i| = \sqrt{49 + 16}$ and $|-3 + 2i| = \sqrt{9 + 4}$.

Example 5

Prove the triangle inequality (a) geometrically, using the parallelogram, and (b) algebraically, using $|z|^2 = z\bar{z}$.

(a) Let A represent z_1 and C represent $z_1 + z_2$, so $AC = |z_2|$. In $\triangle OAC$, $OC \leq OA + AC$ because one side of a triangle is at most the sum of the other two. That is $|z_1 + z_2| \leq |z_1| + |z_2|$.

(b)

$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1z_2 \\&= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) \\&\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| = (|z_1| + |z_2|)^2.\end{aligned}$$

Both sides are non-negative, so take square roots.

In (b), $z_1\bar{z}_2 + \bar{z}_1z_2$ is a number plus its conjugate, so it equals $2\operatorname{Re}(z_1\bar{z}_2)$. Then use $\operatorname{Re}(w) \leq |w|$ and $|z_1\bar{z}_2| = |z_1||z_2|$. In (a), the triangle is degenerate (equality) when O, A, C are collinear with A between O and C .

Example 6(a)

Given $|z| < \frac{1}{2}$, show that $|(1+i)z^3 + iz| < \frac{3}{4}$.

$$|(1+i)z^3 + iz| \leq |1+i||z|^3 + |i||z| = \sqrt{2}|z|^3 + |z| < \frac{\sqrt{2}}{8} + \frac{1}{2} < \frac{3}{4}.$$

Apply the triangle inequality, then $|z_1z_2| = |z_1||z_2|$. Since $\frac{\sqrt{2}}{8} \approx 0.177$, the bound is about $0.677 < 0.75$.

Example 6(b)

z moves on the circle $|z| = 1$. Find the greatest value of $|z+2| + |z-2|$.

The greatest value is $2\sqrt{5}$, at $z = \pm i$.

With $z = \cos \theta + i \sin \theta$, $|z \pm 2| = \sqrt{5 \pm 4 \cos \theta}$. Then $(\sqrt{5+4c} + \sqrt{5-4c})^2 = 10 + 2\sqrt{25-16c^2}$, which is greatest when $c = \cos \theta = 0$, giving 20 and so $2\sqrt{5}$. The triangle inequality alone only gives the bound 6, which is never reached: a good point to discuss.

HSC practice • Adding unit vectors

Let $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $w = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. (i) Write z and w in polar form. (ii) Show that $|z + w|^2 = \frac{4 + \sqrt{2} - \sqrt{6}}{2}$. (iii) A and C represent z and $z + w$. Explain why $\angle AOC = \frac{7\pi}{24}$, and hence find $\cos \frac{7\pi}{24}$.

(i) $z = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$ and $w = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$. (ii) $|z + w|^2 = 2 + 2 \cos \frac{7\pi}{12} = 2 + \frac{\sqrt{2} - \sqrt{6}}{2} = \frac{4 + \sqrt{2} - \sqrt{6}}{2}$. (iii) $|z| = |w| = 1$, so $OACB$ is a rhombus (with B representing w) and the diagonal OC bisects $\angle AOB = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$, giving $\angle AOC = \frac{7\pi}{24}$. In $\triangle OAC$ the perpendicular from A meets OC at its midpoint, so $\cos \frac{7\pi}{24} = \frac{|z + w|/2}{1} = \frac{\sqrt{8 + 2\sqrt{2} - 2\sqrt{6}}}{4} \approx 0.6088$.
(ii) $|z + w|^2 = |z|^2 + |w|^2 + 2 \operatorname{Re}(z\bar{w})$, and $z\bar{w} = \operatorname{cis}(\frac{\pi}{6} - \frac{3\pi}{4})$ has real part $\cos \frac{7\pi}{12} = \frac{\sqrt{2} - \sqrt{6}}{4}$. (iii) The diagonals of a rhombus bisect its angles and meet at right angles, so the foot of the perpendicular from A is the midpoint of OC . Then $\cos \frac{7\pi}{24} = \frac{|z + w|}{2} = \frac{1}{2} \sqrt{\frac{4 + \sqrt{2} - \sqrt{6}}{2}} = \frac{\sqrt{8 + 2\sqrt{2} - 2\sqrt{6}}}{4}$.
(The original paper uses the exponential form.)

Exit ticket

- Z is $2 + i$ and W is $-1 + 5i$. Find the complex number represented by $\overrightarrow{WZ}$ and the length WZ .
 $\overrightarrow{WZ} = 3 - 4i$ and $WZ = 5$.
Head minus tail: $z - w = (2 + i) - (-1 + 5i) = 3 - 4i$, and $|3 - 4i| = \sqrt{9 + 16} = 5$.
- Explain geometrically why $|z + w| = |z - w|$ exactly when $OZ \perp OW$.
 $z + w$ and $z - w$ are the two diagonals of the parallelogram on OZ and OW . The diagonals are equal exactly when the parallelogram is a rectangle, that is, when $OZ \perp OW$.
A parallelogram with equal diagonals is a rectangle, so adjacent sides are perpendicular. This idea is used again in Lesson 8.

Practice worksheet

Vectors and parallelograms

- P , Q and R represent $1 + 4i$, $-2 + i$ and $3 - 2i$. Find the complex numbers represented by (a) $\overrightarrow{PQ}$, (b) $\overrightarrow{QR}$, (c) $\overrightarrow{RP}$, and show that $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = 0$.
(a) $q - p = -3 - 3i$. (b) $r - q = 5 - 3i$. (c) $p - r = -2 + 6i$. Sum: $(-3 + 5 - 2) + (-3 - 3 + 6)i = 0$.
- $OABC$ is a parallelogram with A at $6 - i$ and C at $-1 + 4i$. Find B , $\overrightarrow{AC}$ and the lengths of the two diagonals.
 $B = 5 + 3i$. $\overrightarrow{AC} = c - a = -7 + 5i$. $OB = \sqrt{34}$ and $AC = \sqrt{74}$.

Distance and direction

3. W is $2 + 3i$ and Z is $5 + 7i$. Find $|z - w|$ and sketch $\overrightarrow{WZ}$ on an Argand diagram. Find $\arg(z - w)$ correct to two decimal places.

$z - w = 3 + 4i$, so $|z - w| = 5$ and $\arg(z - w) = \tan^{-1} \frac{4}{3} \approx 0.93$ radians.

Triangle inequality

4. Show that for any complex numbers $|z_1 - z_2| \geq ||z_1| - |z_2||$.

$|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$, so $|z_1| - |z_2| \leq |z_1 - z_2|$. Swapping the roles gives $|z_2| - |z_1| \leq |z_1 - z_2|$. Together these give the result.

5. Given $|z| \leq 2$, find the greatest possible value of $|z^2 + 3z|$ using the triangle inequality, and explain why it is actually attained.

$|z^2 + 3z| \leq |z|^2 + 3|z| \leq 4 + 6 = 10$. At $z = 2$, $|4 + 6| = 10$, so the bound is attained.

6. Prove that $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$, and interpret it for a parallelogram.

$|z + w|^2 + |z - w|^2 = (|z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w})) + (|z|^2 + |w|^2 - 2\operatorname{Re}(z\bar{w})) = 2|z|^2 + 2|w|^2$.
In a parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the four sides.

HSC-style

7. Adapted HSC-style. Let $z = \operatorname{cis} \frac{\pi}{3}$ and $w = \operatorname{cis} \frac{\pi}{2}$. (a) Show that $|z + w| = 2 \cos \frac{\pi}{12}$. (b) Hence find $\arg(z + w)$ and explain your answer using a rhombus. (c) By equating real parts, show that $2 \cos \frac{\pi}{12} \cos \frac{5\pi}{12} = \frac{1}{2}$.

(a) $|z + w|^2 = 2 + 2 \cos \left(\frac{\pi}{3} - \frac{\pi}{2} \right) = 2 + 2 \cos \frac{\pi}{6} = 2 + \sqrt{3}$, and $\left(2 \cos \frac{\pi}{12} \right)^2 = 2(1 + \cos \frac{\pi}{6}) = 2 + \sqrt{3}$, so $|z + w| = 2 \cos \frac{\pi}{12}$. (b) $|z| = |w| = 1$ gives a rhombus, so $z + w$ bisects the angle: $\arg(z + w) = \frac{5\pi}{12}$. (c) $\operatorname{Re}(z + w) = \cos \frac{\pi}{3} + \cos \frac{\pi}{2} = \frac{1}{2}$, and also $\operatorname{Re}(z + w) = |z + w| \cos \arg(z + w) = 2 \cos \frac{\pi}{12} \cos \frac{5\pi}{12}$.