

Worked solutions

Lesson 6 • Multiplying and dividing in polar form

Warm-up • Retrieval from Lesson 5

- (a) Write $1 - i$ in polar form.

$$\sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$$

$r = \sqrt{2}$. The point is in the fourth quadrant with reference angle $\frac{\pi}{4}$, so $\theta = -\frac{\pi}{4}$.

- (b) State $\text{Arg}(-2)$.

π (not $-\pi$)

The principal argument lies in $(-\pi, \pi]$, which includes π and excludes $-\pi$.

- (c) Expand $\cos(A + B)$ and $\sin(A + B)$.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \text{ and } \sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Keep these on the board: they are exactly what Example 1 needs.

- (d) Write $2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$ as $x + iy$.

$$-\sqrt{3} + i$$

$$2 \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = -\sqrt{3} + i.$$

Example 1 • Proving the multiplication rule

Prove that

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)). \end{aligned}$$

Expand using $i^2 = -1$, collect real and imaginary parts, then recognise the compound-angle formulas for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. So $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, up to a multiple of 2π : multiply the moduli and add the arguments.

Think

In words: what does multiplying z by $r \operatorname{cis} \theta$ do to the point z on the Argand diagram?

It multiplies the distance from O by r (a dilation about O) and adds θ to the argument (a rotation about O through θ).

This is the geometric meaning of “multiply the moduli, add the arguments”. It is developed fully in Lesson 8.

Example 2 • Multiplying in polar form

(a) $4 \operatorname{cis} \frac{3\pi}{4} \times 3 \operatorname{cis} \frac{\pi}{2}$

$$12 \operatorname{cis} \left(-\frac{3\pi}{4} \right)$$

Multiply moduli: $4 \times 3 = 12$. Add arguments: $\frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$. This is more than π , so subtract 2π to get $-\frac{3\pi}{4}$.

(b) $\left(2 \operatorname{cis} \frac{2\pi}{3} \right)^2$

$$4 \operatorname{cis} \left(-\frac{2\pi}{3} \right)$$

$2^2 = 4$ and $\frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$, which is outside $(-\pi, \pi]$, so subtract 2π to get $-\frac{2\pi}{3}$.

Example 3 • Dividing in polar form

(a) $\frac{12 \operatorname{cis} 0}{6 \operatorname{cis} \frac{2\pi}{3}}$

$$2 \operatorname{cis} \left(-\frac{2\pi}{3} \right) = 2 \left(\cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3} \right) = -1 - \sqrt{3}i$$

Divide moduli: $12 \div 6 = 2$. Subtract arguments: $0 - \frac{2\pi}{3} = -\frac{2\pi}{3}$, already in $(-\pi, \pi]$.

(b) $\frac{6 \operatorname{cis} \left(-\frac{\pi}{3} \right)}{2 \operatorname{cis} \frac{2\pi}{3}}$

$$3 \operatorname{cis} \pi = -3$$

$6 \div 2 = 3$ and $-\frac{\pi}{3} - \frac{2\pi}{3} = -\pi$. The principal argument lies in $(-\pi, \pi]$, so write π instead of $-\pi$. A negative real number has principal argument π .

Example 4 • Proving the identities

(a) Prove that $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ and $\arg \left(\frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2$.

Let $w = \frac{z_1}{z_2}$, so $z_1 = wz_2$. By the product rule, $|z_1| = |w||z_2|$ and $\arg z_1 = \arg w + \arg z_2$.

Rearranging gives $|w| = \frac{|z_1|}{|z_2|}$ and $\arg w = \arg z_1 - \arg z_2$.

Use the product rule you have already proved, applied to $z_1 = wz_2$. (Alternatively, multiply top and bottom by $\cos \theta_2 - i \sin \theta_2$ and simplify.)

- (b) Prove that $\arg \bar{z} = -\arg z$ and $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ using polar form.

If $z = r \operatorname{cis} \theta$ then $\bar{z} = r(\cos \theta - i \sin \theta) = r(\cos(-\theta) + i \sin(-\theta))$, so $\arg \bar{z} = -\theta$. Then $\overline{z_1 z_2} = r_1 \operatorname{cis}(-\theta_1) \cdot r_2 \operatorname{cis}(-\theta_2) = r_1 r_2 \operatorname{cis}(-(\theta_1 + \theta_2)) = \bar{z}_1 \bar{z}_2$.

Cosine is even and sine is odd. The conjugate of a product has modulus $r_1 r_2$ and argument $-(\theta_1 + \theta_2)$, which is exactly the product of the conjugates.

Example 5 • Two forms, exact values

- (a) Write z_1 and z_2 in polar form.

$$z_1 = 2 \operatorname{cis} \frac{\pi}{3} \text{ and } z_2 = \sqrt{2} \operatorname{cis} \frac{\pi}{4}.$$

$|z_1| = \sqrt{1+3} = 2$ with $\tan \theta = \sqrt{3}$. $|z_2| = \sqrt{2}$ with $\tan \theta = 1$. Both are in the first quadrant.

- (b) Find $z_1 z_2$ and $\frac{z_1}{z_2}$ in polar form and in Cartesian form.

$$z_1 z_2 = 2\sqrt{2} \operatorname{cis} \frac{7\pi}{12} = (1 - \sqrt{3}) + (1 + \sqrt{3})i. \quad \frac{z_1}{z_2} = \sqrt{2} \operatorname{cis} \frac{\pi}{12} = \frac{(1 + \sqrt{3}) + (\sqrt{3} - 1)i}{2}.$$

Polar: multiply moduli and add arguments ($\frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$), or divide moduli and subtract ($\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$). Cartesian: expand $(1 + \sqrt{3}i)(1 + i)$, and for the quotient multiply top and bottom by $1 - i$.

- (c) Hence find the exact values of $\sin \frac{7\pi}{12}$ and $\cos \frac{\pi}{12}$.

$$\sin \frac{7\pi}{12} = \cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}.$$

Equate imaginary parts of the two forms of $z_1 z_2$: $2\sqrt{2} \sin \frac{7\pi}{12} = 1 + \sqrt{3}$, so $\sin \frac{7\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$. Equate real parts of $\frac{z_1}{z_2}$: $\sqrt{2} \cos \frac{\pi}{12} = \frac{1+\sqrt{3}}{2}$, so $\cos \frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$.

HSC practice • Powers in polar form

Let $z = \sqrt{3} + i$. (i) Write z in polar form. (ii) Hence express z^7 in the form $x + iy$.

$$(i) z = 2 \operatorname{cis} \frac{\pi}{6}. \quad (ii) z^7 = 128 \operatorname{cis} \frac{7\pi}{6} = -64\sqrt{3} - 64i.$$

$$\arg(z^n) = n \arg z \text{ and } |z^n| = |z|^n, \text{ so } z^7 = 2^7 \operatorname{cis} \frac{7\pi}{6} = 128 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right).$$

HSC practice • Powers in polar form

Let $z = -\sqrt{3} + i$. Use polar form to find z^{10} in the form $x + iy$.

$$z = 2 \operatorname{cis} \frac{5\pi}{6} \text{ and } z^{10} = 1024 \operatorname{cis} \frac{25\pi}{3} = 1024 \operatorname{cis} \frac{\pi}{3} = 512 + 512\sqrt{3}i.$$

$10 \times \frac{5\pi}{6} = \frac{25\pi}{3} = 8\pi + \frac{\pi}{3}$. Remove whole turns of 2π , then use $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$. The rule $\arg(z^n) = n \arg z$ is enough here; de Moivre's theorem is proved by induction later in this unit.

HSC practice • Modulus and argument of a new number

Let z be a non-zero complex number and $w = \frac{z^2}{|z|}$. Find $|w|$ and $\arg w$ in terms of $|z|$ and $\arg z$.

Describe where w lies compared with z .

$|w| = |z|$ and $\arg w = 2 \arg z$. So w is on the same circle $|w| = |z|$ about O as z , at double the angle.

$|w| = \frac{|z|^2}{|z|} = |z|$. Dividing by the positive real number $|z|$ does not change the argument (a positive real has argument 0), so $\arg w = \arg(z^2) - 0 = 2 \arg z$.

Exit ticket

1. Use polar form to express $\frac{(1+i)^5}{(1-i)^3}$ in the form $x + iy$.

2

$1+i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$ and $1-i = \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4}\right)$. The quotient is $\frac{(\sqrt{2})^5 \operatorname{cis} \frac{5\pi}{4}}{(\sqrt{2})^3 \operatorname{cis} \left(-\frac{3\pi}{4}\right)} = 2 \operatorname{cis} \left(\frac{5\pi}{4} + \frac{3\pi}{4}\right) = 2 \operatorname{cis} 2\pi = 2$.

2. Prove that $|z^n| = |z|^n$ and $\arg(z^n) = n \arg z$ for $n = 2$ and $n = 3$.

With $z = r \operatorname{cis} \theta$: $z^2 = z \cdot z = r^2 \operatorname{cis} 2\theta$, and $z^3 = z^2 \cdot z = r^3 \operatorname{cis} 3\theta$.

Apply the product rule once for $n = 2$ and again for $n = 3$. The general case is proved by induction in Complex Numbers II (de Moivre's theorem).

Practice worksheet

Multiplying and dividing

1. Express in polar form with the principal argument.

(a) $3 \operatorname{cis} \frac{2\pi}{3} \times 2 \operatorname{cis} \frac{3\pi}{4}$

(b) $\frac{8 \operatorname{cis} \frac{\pi}{6}}{2 \operatorname{cis} \frac{5\pi}{6}}$

(c) $(\sqrt{2} \operatorname{cis} \frac{3\pi}{4})^3$

(a) $6 \operatorname{cis} \frac{17\pi}{12} = 6 \operatorname{cis} \left(-\frac{7\pi}{12}\right)$. (b) $4 \operatorname{cis} \left(-\frac{2\pi}{3}\right)$. (c) $2\sqrt{2} \operatorname{cis} \frac{9\pi}{4} = 2\sqrt{2} \operatorname{cis} \frac{\pi}{4}$.

2. Write $z_1 = \sqrt{3} - i$ and $z_2 = -1 + i$ in polar form, then find $z_1 z_2$ and $\frac{z_1}{z_2}$ in polar form.

$z_1 = 2 \operatorname{cis} \left(-\frac{\pi}{6}\right)$, $z_2 = \sqrt{2} \operatorname{cis} \frac{3\pi}{4}$. $z_1 z_2 = 2\sqrt{2} \operatorname{cis} \frac{7\pi}{12}$. $\frac{z_1}{z_2} = \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{6} - \frac{3\pi}{4}\right) = \sqrt{2} \operatorname{cis} \left(-\frac{11\pi}{12}\right)$.

Exact values

3. Let $z_1 = \sqrt{3} + i$ and $z_2 = 1 - i$. Find $z_1 z_2$ in polar form and in Cartesian form. Hence find the exact value of $\cos \left(-\frac{\pi}{12}\right)$.

$z_1 = 2 \operatorname{cis} \frac{\pi}{6}$, $z_2 = \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4}\right)$, so $z_1 z_2 = 2\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{12}\right)$. **Cartesian:** $(\sqrt{3} + i)(1 - i) = (\sqrt{3} + 1) + (1 - \sqrt{3})i$. Equating real parts, $2\sqrt{2} \cos \frac{\pi}{12} = \sqrt{3} + 1$, so $\cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$.

Proofs

4. Prove that $\arg\left(\frac{1}{z}\right) = -\arg z$ for $z \neq 0$, and explain the connection with the conjugate $\bar{z}$.

With $1 = 1 \operatorname{cis} 0$ and $z = r \operatorname{cis} \theta$, the quotient rule gives $\frac{1}{z} = \frac{1}{r} \operatorname{cis}(0 - \theta)$, so $\arg \frac{1}{z} = -\theta$. Since $\bar{z} = r \operatorname{cis}(-\theta)$ has the same argument, $\frac{1}{z}$ and $\bar{z}$ point in the same direction, consistent with $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

5. Prove that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ using polar form.

$\bar{z}_1 = r_1 \operatorname{cis}(-\theta_1)$ and $\bar{z}_2 = r_2 \operatorname{cis}(-\theta_2)$, so $\frac{\bar{z}_1}{\bar{z}_2} = \frac{r_1}{r_2} \operatorname{cis}(-\theta_1 + \theta_2)$. Also $\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$, whose conjugate is $\frac{r_1}{r_2} \operatorname{cis}-(\theta_1 - \theta_2)$, the same.

Powers

6. Use polar form to find $(1 - i)^8$ and $(-1 + \sqrt{3}i)^6$.

$1 - i = \sqrt{2} \operatorname{cis}(-\frac{\pi}{4})$, so $(1 - i)^8 = 16 \operatorname{cis}(-2\pi) = 16$. $-1 + \sqrt{3}i = 2 \operatorname{cis} \frac{2\pi}{3}$, so the sixth power is $64 \operatorname{cis} 4\pi = 64$.

HSC-style

7. Let $z = 1 + i$ and $w = \sqrt{3} + i$. (a) Write z and w in polar form. (b) Find $\frac{w^6}{z^4}$. (c) Find the smallest positive integer n such that $\left(\frac{w}{z}\right)^n$ is real.

(a) $z = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$, $w = 2 \operatorname{cis} \frac{\pi}{6}$. (b) $w^6 = 64 \operatorname{cis} \pi$ and $z^4 = 4 \operatorname{cis} \pi$, so $\frac{w^6}{z^4} = 16 \operatorname{cis} 0 = 16$. (c) $\frac{w}{z} = \sqrt{2} \operatorname{cis}(-\frac{\pi}{12})$, so $\left(\frac{w}{z}\right)^n$ has argument $-\frac{n\pi}{12}$, which is a multiple of π first when $n = 12$.