

Worked solutions

Lesson 5 • Modulus–argument form

Warm-up • Exact values

(a) $\tan^{-1} \sqrt{3}$

$\frac{\pi}{3}$

 $\tan \frac{\pi}{3} = \sqrt{3}$, and $\frac{\pi}{3}$ lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(b) $\cos \frac{2\pi}{3}$

$-\frac{1}{2}$

 $\frac{2\pi}{3}$ is in the second quadrant, where cosine is negative, with reference angle $\frac{\pi}{3}$.

(c) $\sin \left(-\frac{3\pi}{4} \right)$

$-\frac{1}{\sqrt{2}}$

 $-\frac{3\pi}{4}$ is in the third quadrant, where sine is negative, with reference angle $\frac{\pi}{4}$.

(d) Find $\theta \in (-\pi, \pi]$ with $\cos \theta = -\frac{1}{2}$ and $\sin \theta = -\frac{\sqrt{3}}{2}$.

$\theta = -\frac{2\pi}{3}$

Both cosine and sine are negative, so θ is in the third quadrant. The reference angle is $\frac{\pi}{3}$, so $\theta = -(\pi - \frac{\pi}{3}) = -\frac{2\pi}{3}$.

Discuss

(a) Sketch: mark r and θ for a point $z = x + iy$ in each quadrant.

In each quadrant, r is the length of the line from O to z and θ is the angle from the positive real axis to that line, anticlockwise positive and clockwise negative. For example, $3 + 4i$ has $r = 5$ and θ marked above the axis, while $3 - 4i$ has the same r and $-\theta$ below it.

The sketch shows why the quadrant matters: $\cos \theta = \frac{x}{r}$ and $\sin \theta = \frac{y}{r}$ have signs that depend on where the point is.

(b) Write three different values of $\arg(1 + i)$. Which one is $\text{Arg}(1 + i)$?

For example $\frac{\pi}{4}$, $\frac{9\pi}{4}$ and $-\frac{7\pi}{4}$ (any $\frac{\pi}{4} + 2k\pi$). $\text{Arg}(1 + i) = \frac{\pi}{4}$.

Adding or subtracting 2π gives the same point, so $\arg$ has many values. Only $\frac{\pi}{4}$ lies in $(-\pi, \pi]$.

Example 1 • Cartesian to polar

Plot $z = 3 + 4i$. Find $|z|$ and $\text{Arg } z$, and write z in modulus-argument form. Repeat for $\bar{z} = 3 - 4i$.

$|z| = 5$ and $\text{Arg } z = \tan^{-1} \frac{4}{3} \approx 0.93$, so $z \approx 5(\cos 0.93 + i \sin 0.93)$. Also $\bar{z} \approx 5(\cos(-0.93) + i \sin(-0.93))$.

$r = \sqrt{3^2 + 4^2} = 5$. The point is in the first quadrant, so $\theta = \tan^{-1} \frac{4}{3} \approx 0.93$ radians (about 53°).

The conjugate is the reflection in the real axis, so it has the same modulus and the opposite argument.

Example 2 • Second and third quadrants

(a) $-1 + i$

$$\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

$r = \sqrt{2}$. The point is in the second quadrant with reference angle $\frac{\pi}{4}$, so $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

(b) $-1 - \sqrt{3}i$

$$2 \left(\cos \left(-\frac{2\pi}{3} \right) + i \sin \left(-\frac{2\pi}{3} \right) \right)$$

$r = \sqrt{1 + 3} = 2$. The point is in the third quadrant with reference angle $\frac{\pi}{3}$, so $\theta = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$.

Example 3 • Choosing the quadrant

(a) Find $\text{Arg}(\sqrt{6} - \sqrt{2}i)$.

$$-\frac{\pi}{6}$$

The point is in the fourth quadrant. $\tan \alpha = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$, so $\alpha = \frac{\pi}{6}$ and $\text{Arg} = -\frac{\pi}{6}$. (The modulus is $\sqrt{8} = 2\sqrt{2}$.)

Example 3 • Choosing the quadrant

(b) Write $-4 - 3i$ in polar form, with the argument correct to 2 decimal places.

$$5(\cos(-2.50) + i \sin(-2.50))$$

$r = 5$ (a 3-4-5 triangle). The point is in the third quadrant, with $\alpha = \tan^{-1} \frac{3}{4} \approx 0.64$. So $\text{Arg} = -\pi + 0.64 \approx -2.50$.

Example 4 • On the axes

(a) $5i$

$$5 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

The point is on the positive imaginary axis, a quarter-turn from the real axis.

(b) -4

$$4(\cos \pi + i \sin \pi)$$

$\text{Arg}(-4) = \pi$, not $-\pi$: the interval $(-\pi, \pi]$ includes π and excludes $-\pi$.

(c) $-\frac{i}{2}$

$$\frac{1}{2} \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right)$$

The point is on the negative imaginary axis, a quarter-turn clockwise from the real axis.

Example 5 • Polar to Cartesian

(a) $4 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

$$2 + 2\sqrt{3}i$$

$$4 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right).$$

(b) $10 \text{ cis } \frac{2\pi}{3}$

$$-5 + 5\sqrt{3}i$$

$$10 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right).$$

(c) $\sqrt{6} \text{ cis } \left(-\frac{3\pi}{4} \right)$

$$-\sqrt{3} - \sqrt{3}i$$

$$\sqrt{6} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = -\sqrt{3} - \sqrt{3}i.$$

(d) $2 \text{ cis } \pi$

$$-2$$

$$2(-1 + 0i) = -2.$$

Example 6 • Conjugate and negative in polar form

(a) $\bar{z}$

$$2 \text{ cis } \left(-\frac{\pi}{6} \right)$$

Negate the argument.

(b) $-z$

$$2 \text{ cis } \left(-\frac{5\pi}{6} \right)$$

$$\frac{\pi}{6} - \pi = -\frac{5\pi}{6}, \text{ which lies in } (-\pi, \pi].$$

(c) $-\bar{z}$

$$2 \text{ cis } \frac{5\pi}{6}$$

$$-\frac{\pi}{6} + \pi = \frac{5\pi}{6}.$$

(d) Plot all four points.

$z = \sqrt{3} + i, \bar{z} = \sqrt{3} - i, -z = -\sqrt{3} - i$ and $-\bar{z} = -\sqrt{3} + i$: the vertices of a rectangle centred at the origin.

All four have modulus 2, so they lie on a circle of radius 2.

HSC practice • Polar form

Let $z = -\sqrt{3} + i$. (i) Find $|z|$ and $\text{Arg } z$. (ii) Write z in the form $r(\cos \theta + i \sin \theta)$.

(i) $|z| = 2$ and $\text{Arg } z = \frac{5\pi}{6}$. (ii) $z = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$.

$|z| = \sqrt{3+1} = 2$. The point is in the second quadrant with $\tan \alpha = \frac{1}{\sqrt{3}}$, so $\alpha = \frac{\pi}{6}$ and $\text{Arg } z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Spot the error • Arguments

(a) $z = -2 + 2i$: $\text{Arg } z = \tan^{-1} \left(\frac{2}{-2} \right) = -\frac{\pi}{4}$

$$\text{Arg } z = \frac{3\pi}{4}$$

The point $-2 + 2i$ is in the second quadrant, but $\tan^{-1}$ only returns values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. The reference angle is $\frac{\pi}{4}$, so $\text{Arg } z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. This is why the method says to sketch first.

(b) $\text{Arg}(-4) = -\pi$

$$\text{Arg}(-4) = \pi$$

The principal argument lies in $(-\pi, \pi]$, which includes π and excludes $-\pi$.

(c) $1 - \sqrt{3}i = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

$$1 - \sqrt{3}i = 2 \left(\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right)$$

The point is in the fourth quadrant (positive real part, negative imaginary part), so the argument is $-\frac{\pi}{3}$. The written form $2 \text{cis } \frac{\pi}{3}$ is the point $1 + \sqrt{3}i$.

(d) $2 \left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right)$ has argument $\frac{\pi}{6}$

The argument is $-\frac{\pi}{6}$.

The expression is not in polar form because of the minus sign. Since cosine is even and sine is odd, $\cos \frac{\pi}{6} - i \sin \frac{\pi}{6} = \cos \left(-\frac{\pi}{6} \right) + i \sin \left(-\frac{\pi}{6} \right)$.

Exit ticket

1. Write $-2\sqrt{3} + 2i$ in polar form.

$$4 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$$

$r = \sqrt{12+4} = 4$. The point is in the second quadrant, with $\alpha = \tan^{-1} \frac{2}{2\sqrt{3}} = \frac{\pi}{6}$, so $\theta = \frac{5\pi}{6}$.

2. Express $6 \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$ as $x + iy$.

$$3\sqrt{2} - 3\sqrt{2}i$$

$$6 \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = \frac{6}{\sqrt{2}} - \frac{6}{\sqrt{2}}i \text{ and } \frac{6}{\sqrt{2}} = 3\sqrt{2}.$$

3. Explain why $\text{Arg}(-5) = \pi$ and not $-\pi$.

The principal argument lies in $(-\pi, \pi]$: π is included and $-\pi$ is not.

$\arg(-5)$ has the values $\pi + 2k\pi$, including both π and $-\pi$. Exactly one value must be chosen as Arg , and the interval $(-\pi, \pi]$ is closed at π and open at $-\pi$.

Practice worksheet

Cartesian to polar

1. Write in modulus-argument form with the principal argument in exact form.

(a) $1 + i$

(b) $-\sqrt{3} + i$

(c) $-2 - 2i$

(d) $\sqrt{3} - 3i$

(a) $\sqrt{2} \text{ cis } \frac{\pi}{4}$. (b) $2 \text{ cis } \frac{5\pi}{6}$. (c) $r = 2\sqrt{2}$, third quadrant: $2\sqrt{2} \text{ cis } \left(-\frac{3\pi}{4}\right)$. (d) $r = \sqrt{3+9} = 2\sqrt{3}$, fourth quadrant with $\tan \alpha = \sqrt{3}$: $2\sqrt{3} \text{ cis } \left(-\frac{\pi}{3}\right)$.

2. Write in modulus-argument form.

(a) -7

(b) $-3i$

(c) 12

(a) $7(\cos \pi + i \sin \pi)$. (b) $3 \left(\cos \left(-\frac{\pi}{2}\right) + i \sin \left(-\frac{\pi}{2}\right) \right)$. (c) $12(\cos 0 + i \sin 0)$.

3. Write $-5 + 12i$ in polar form with the argument correct to 2 decimal places.

$r = 13$. Second quadrant: $\text{Arg} = \pi - \tan^{-1} \frac{12}{5} \approx 3.1416 - 1.1760 = 1.97$. So $z \approx 13(\cos 1.97 + i \sin 1.97)$.

Polar to Cartesian

4. Express in the form $x + iy$.

(a) $2 \text{ cis } \frac{\pi}{6}$

(b) $8 \text{ cis } \left(-\frac{\pi}{2}\right)$

(c) $\sqrt{8} \text{ cis } \frac{3\pi}{4}$

(d) $4 \text{ cis } \left(-\frac{2\pi}{3}\right)$

(a) $\sqrt{3} + i$. (b) $-8i$. (c) $2\sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -2 + 2i$. (d) $4 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -2 - 2\sqrt{3}i$.

Conjugate and negative

5. Let $z = 3 \text{ cis } \frac{3\pi}{4}$. Write $\bar{z}$, $-z$ and $-\bar{z}$ in polar form.

$\bar{z} = 3 \text{ cis } \left(-\frac{3\pi}{4}\right)$. $-z$: $\frac{3\pi}{4} - \pi = -\frac{\pi}{4}$, so $-z = 3 \text{ cis } \left(-\frac{\pi}{4}\right)$. $-\bar{z}$: $-\frac{3\pi}{4} + \pi = \frac{\pi}{4}$, so $-\bar{z} = 3 \text{ cis } \frac{\pi}{4}$.

6. Explain why $\text{Arg}(-5) = \pi$ and not $-\pi$. What is $\text{Arg}(\overline{-5})$?

The principal argument lies in $(-\pi, \pi]$: π is included and $-\pi$ is excluded. Since -5 is real, $\overline{-5} = -5$, so $\text{Arg} = \pi$ again. (The rule $\text{Arg } \bar{z} = -\text{Arg } z$ fails for negative real numbers.)

HSC-style

7. Let $z = 1 - i\sqrt{3}$. (i) Find $|z|$ and $\text{Arg } z$. (ii) Write z in the form $r(\cos \theta + i \sin \theta)$. (iii) Write $-z$ in polar form with the principal argument.

(i) $|z| = \sqrt{1 + 3} = 2$. The point is in the fourth quadrant with $\tan \alpha = \sqrt{3}$, so $\text{Arg } z = -\frac{\pi}{3}$. (ii) $z = 2 \left(\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right)$. (iii) $-z = 2 \text{ cis } \left(-\frac{\pi}{3} + \pi \right) = 2 \text{ cis } \frac{2\pi}{3}$.