

Worked solutions

Lesson 4 • The Argand diagram

Warm-up • Retrieval from Lessons 1 to 3

- (a) Find real a, b with $(a + ib)(1 + i) = 3 + i$.

$$a = 2, b = -1$$

$$a + ib = \frac{3 + i}{1 + i} = \frac{(3 + i)(1 - i)}{2} = \frac{4 - 2i}{2} = 2 - i. \text{ Check: } (2 - i)(1 + i) = 2 + 2i - i + 1 = 3 + i.$$

- (b) Find the square roots of $2i$.

$$\pm(1 + i)$$

$$(1 + i)^2 = 1 + 2i - 1 = 2i, \text{ so the square roots are } \pm(1 + i).$$

- (c) Find a real quadratic with root $1 + 2i$.

$$z^2 - 2z + 5 = 0$$

$$\text{The other root is } 1 - 2i. \text{ Sum} = 2, \text{ product} = 1 + 4 = 5.$$

- (d) Find the distance between $(1, 2)$ and $(4, 6)$.

$$5$$

$$\sqrt{3^2 + 4^2} = 5. \text{ This previews distance on the Argand diagram.}$$

Check • Reading the plane

- (a) Write the point for $-3 + 5i$.

$$(-3, 5).$$

The real part is the x -coordinate and the imaginary part is the y -coordinate.

- (b) Write the point for $2i$.

$$(0, 2).$$

$$2i = 0 + 2i, \text{ so it lies on the imaginary axis.}$$

- (c) Write the point for 7 .

$$(7, 0).$$

$$7 = 7 + 0i, \text{ so it lies on the real axis.}$$

- (d) Write the point for $(2 + i)i$.

$$(-1, 2).$$

$$(2 + i)i = 2i + i^2 = -1 + 2i. \text{ This previews multiplying by } i.$$

(e) Write the number for $(-3, 0)$.

-3 .

A point on the real axis is a real number.

(f) Write the number for $(0, 3)$.

$3i$.

A point on the imaginary axis is purely imaginary.

(g) Write the number for $(7, -5)$.

$7 - 5i$.

$x = 7$ and $y = -5$, so $z = 7 + (-5)i$.

(h) Write the number for (a, b) .

$a + ib$.

This is the general rule: (x, y) represents $x + iy$.

Example 1

Let $z = 3 + 4i$. Plot (a) z , (b) $\bar{z}$, (c) $-z$, (d) iz . Describe each as a transformation of z .

$z = (3, 4)$; $\bar{z} = 3 - 4i$ at $(3, -4)$; $-z = -3 - 4i$ at $(-3, -4)$; $iz = -4 + 3i$ at $(-4, 3)$.

$\bar{z}$ is the reflection of z in the real axis. $-z$ is a rotation of 180° about the origin. $iz = i(3 + 4i) = -4 + 3i$ is a rotation of 90° anticlockwise about the origin. In general $iz = i(x + iy) = -y + ix$.

The gradient of Oz is $\frac{4}{3}$ and the gradient of $O(iz)$ is $-\frac{3}{4}$: the product is -1 , so they are perpendicular.

Example 2 • Powers of i on the plane

(a) Plot the four points.

$A = 1 + 3i$, $B = -3 + i$, $C = -1 - 3i$, $D = 3 - i$.

$iz = i + 3i^2 = -3 + i$, $i^2z = -z = -1 - 3i$, $i^3z = -iz = 3 - i$.

(b) What type of quadrilateral is $ABCD$?

A square centred at the origin.

All sides have length $\sqrt{20}$ (for example AB : $(4, 2)$ has length $\sqrt{20}$) and the diagonals AC and BD are equal, both of length $2\sqrt{10}$.

(c) What does multiplying by i appear to do?

It rotates the point 90° anticlockwise about the origin.

Each step $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ is a quarter-turn. Four quarter-turns return to the start, which is why $i^4 = 1$.

Example 3 • Why multiplying by i is a rotation

- (a) By considering gradients, show that $OA \perp OB$.

$w = i(a + ib) = -b + ia$, so $A = (a, b)$ and $B = (-b, a)$. The gradient of OA is $\frac{b}{a}$ and of OB is $\frac{a}{-b}$. The product is $\frac{b}{a} \times \left(-\frac{a}{b}\right) = -1$, so $OA \perp OB$.

Two lines are perpendicular when the product of their gradients is -1 . (If a or b is 0, the point lies on an axis and the result is clear from a sketch.)

- (b) Use the distance formula to show that $OA = OB$. What type of triangle is $\triangle OAB$?

$OA = \sqrt{a^2 + b^2}$ and $OB = \sqrt{b^2 + a^2}$, so $OA = OB$. The triangle OAB is right-angled and isosceles.

So B is A rotated 90° anticlockwise about O . This is the geometric fact behind rotations in Lesson 8.

Example 4 • Lines and regions

- (a) Sketch $\operatorname{Re}(z) = 5$ and $\operatorname{Im}(z) = -2$.

A vertical line through $(5, 0)$ and a horizontal line through $(0, -2)$.

They meet at the point $5 - 2i$.

- (b) Shade the region where $\operatorname{Im}(z) < 1$ and $\operatorname{Re}(z) \geq -2$.

The region below the dashed line $y = 1$ and to the right of the solid line $x = -2$. The corner $(-2, 1)$ is not included.

Both conditions must hold, so shade the overlap. The corner is excluded because $\operatorname{Im}(z) < 1$ is strict there.

Example 5

Given that $z\bar{z} = 4$, find the locus of z and sketch it.

The circle with centre O and radius 2: $x^2 + y^2 = 4$.

$(x + iy)(x - iy) = x^2 + y^2 = 4$. Since $z\bar{z} = |z|^2$, this is the same as $|z| = 2$.

Example 6 • A locus with fractions

Given that $\frac{1}{z} + \frac{1}{\bar{z}} = 1$, find the locus of z and sketch it.

The circle $(x - 1)^2 + y^2 = 1$, centre $(1, 0)$ and radius 1, excluding the origin.

The left side is undefined at $z = 0$, so $z \neq 0$. Multiply by $z\bar{z}$: $\bar{z} + z = z\bar{z}$, so $2x = x^2 + y^2$ and $(x - 1)^2 + y^2 = 1$. Draw an open circle at the origin. (Alternatively, $\frac{1}{z} + \frac{1}{\bar{z}} = 2 \operatorname{Re}\left(\frac{1}{z}\right)$ by Lesson 2.)

Example 7 • Show that it is a circle

Show that $(z - 2)(\bar{z} - 2) = 9$ represents a circle in the Argand diagram, and sketch it.

It is the circle with centre $(2, 0)$ and radius 3.

Let $z = x + iy$: $(x - 2 + iy)(x - 2 - iy) = (x - 2)^2 + y^2 = 9$. Neater: $\bar{z} - 2 = \overline{z - 2}$, so the left side is $|z - 2|^2$ and the equation says $|z - 2| = 3$: the distance from z to 2 is 3.

Exam practice • Multiple choice and the unit circle

Which equation describes the circle with centre O and radius 2?

(A) $z\bar{z} = 2$

(B) $(z + \bar{z})^2 - (z - \bar{z})^2 = 16$

(C) $\operatorname{Re}(z) + \operatorname{Im}(z) = 2$

(D) $|z|^2 = 2$

(B)

$(z + \bar{z})^2 - (z - \bar{z})^2 = (2x)^2 - (2iy)^2 = 4x^2 + 4y^2 = 16$, so $x^2 + y^2 = 4$. (A) and (D) both give radius $\sqrt{2}$. (C) is a straight line.

Exam practice • Multiple choice and the unit circle

The point Z lies on the unit circle in the first quadrant, as shown. Where does $W = \frac{1}{Z}$ lie?

On the unit circle in the fourth quadrant: W is the reflection of Z in the real axis.

$|Z| = 1$, so $Z\bar{Z} = 1$ and $\frac{1}{Z} = \bar{Z}$. This uses $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ from Lesson 2.

Exit ticket

1. Let $z = -2 + i$. Plot z , $\bar{z}$ and iz on one Argand diagram.

$z = (-2, 1)$, $\bar{z} = (-2, -1)$ and $iz = (-1, -2)$.

$iz = i(-2 + i) = -2i + i^2 = -1 - 2i$, which is z rotated 90° anticlockwise about the origin.

2. Find and sketch the locus of z if $\operatorname{Im}(z^2) = 0$.

Both axes: the lines $x = 0$ and $y = 0$ (the origin is included).

$z^2 = x^2 - y^2 + 2ixy$, so $\operatorname{Im}(z^2) = 2xy = 0$, giving $x = 0$ or $y = 0$.

Practice worksheet

Plotting and transformations

1. Let $z = -4 + 2i$. Plot z , $\bar{z}$, $-z$ and iz on one Argand diagram and describe each transformation.

$z = (-4, 2)$; $\bar{z} = -4 - 2i$ at $(-4, -2)$ (reflection in the real axis); $-z = 4 - 2i$ at $(4, -2)$

(rotation of 180° about O); $iz = -4i + 2i^2 = -2 - 4i$ at $(-2, -4)$ (rotation of 90° anticlockwise about O).

2. Let $z = 2 + i$. Plot z , iz , i^2z and i^3z and name the shape they form.

$z = 2 + i$, $iz = -1 + 2i$, $i^2z = -2 - i$, $i^3z = 1 - 2i$. They are the vertices of a square centred at the origin.

3. Show that if $z = a + ib$ then z , iz and O form a right-angled isosceles triangle.

$iz = -b + ia$. $|z| = \sqrt{a^2 + b^2} = |iz|$, so two sides are equal. The gradients of Oz and $O(iz)$ are $\frac{b}{a}$ and $-\frac{a}{b}$, whose product is -1 , so the angle at O is a right angle.

Lines, regions and loci

4. Sketch each region.

(a) $\operatorname{Re}(z) > 1$

(b) $-1 \leq \operatorname{Im}(z) < 2$

(c) $\operatorname{Re}(z) \leq 3$ and $\operatorname{Im}(z) \geq 0$

(a) The region to the right of the dashed line $x = 1$. (b) The horizontal strip between the solid line $y = -1$ and the dashed line $y = 2$. (c) The region to the left of the solid line $x = 3$ and on or above the real axis (both boundaries solid).

5. Find and sketch the locus of z if (a) $z\bar{z} = 9$, (b) $(z + 1)(\bar{z} + 1) = 4$.

(a) $x^2 + y^2 = 9$: the circle with centre O and radius 3. (b) $(x + 1)^2 + y^2 = 4$: the circle with centre $(-1, 0)$ and radius 2.

6. Show that $z + \bar{z} = 2$ is a line, and find its Cartesian equation.

$z + \bar{z} = 2x = 2$, so $x = 1$: the vertical line $\operatorname{Re}(z) = 1$.

HSC-style

7. Given that $\frac{1}{z} + \frac{1}{\bar{z}} = 2$, find the locus of z and sketch it, clearly indicating any excluded points.

$z \neq 0$. Multiplying by $z\bar{z}$: $\bar{z} + z = 2z\bar{z}$, so $2x = 2(x^2 + y^2)$, that is $x^2 - x + y^2 = 0$ and $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$. This is the circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$, with the origin excluded (open circle at O).