

Worked solutions

Lesson 3 • Quadratic equations and square roots

Warm-up • Retrieval from Lessons 1 and 2

- (a) Simplify i^{103} .

$$-i$$

$$103 = 4 \times 25 + 3, \text{ so } i^{103} = i^3 = -i.$$

- (b) Expand $(3 - 2i)^2$.

$$5 - 12i$$

$$9 - 12i + 4i^2 = 9 - 12i - 4 = 5 - 12i. \text{ This expansion step is used for square roots later in the lesson.}$$

- (c) Express $\frac{2+i}{1-i}$ in the form $a + ib$.

$$\frac{1+3i}{2}$$

$$\text{Multiply by the conjugate of the denominator: } \frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{2+2i+i-1}{2} = \frac{1+3i}{2}.$$

- (d) Solve $x^2 + 16 = 0$.

$$x = \pm 4i$$

$$x^2 = -16 = 16i^2, \text{ so } x = \pm 4i.$$

Example 1 • Quadratic formula

- (a) Find the sum and the product of the roots.

$$\text{Sum} = -2, \text{ product} = 4.$$

$$\text{For } x^2 + 2x + 4 = 0, \text{ the sum of the roots is } -\frac{b}{a} = -2 \text{ and the product is } \frac{c}{a} = 4.$$

- (b) Use the quadratic formula to solve for x .

$$x = -1 \pm i\sqrt{3}$$

$$x = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i. \text{ Check: the sum is } -2 \text{ and the product is } 1 + 3 = 4. \text{ Notice that the two roots are conjugates of each other.}$$

Example 2 • Completing the square

Solve $z^2 + 8z + 25 = 0$ by completing the square.

$$z = -4 \pm 3i$$

$(z + 4)^2 + 9 = 0$, so $(z + 4)^2 = -9$, $z + 4 = \pm 3i$ and $z = -4 \pm 3i$. Note that a sum of squares factorises over $\mathbb{C}$: $z^2 + \lambda^2 = (z + i\lambda)(z - i\lambda)$.

Example 3

Find a quadratic equation with real coefficients that has $w = 2 - i$ as one root.

$$z^2 - 4z + 5 = 0$$

The other root is $\bar{w} = 2 + i$. Sum = 4, product = $4 + 1 = 5$. Equivalently $(z - (2 - i))(z - (2 + i)) = (z - 2)^2 + 1 = 0$.

HSC-style • Coefficients from a root

The equation $z^2 + pz + q = 0$, where p and q are real, has $3 + i$ as a root. Find p and q .

$$p = -6 \text{ and } q = 10.$$

The other root is $3 - i$. Sum of roots = $6 = -p$, so $p = -6$. Product = $(3 + i)(3 - i) = 10 = q$. A common slip is forgetting the minus sign in $-p$ and giving $p = 6$.

Example 4

Find the square roots of -9 .

$$\pm 3i$$

$(\pm 3i)^2 = 9i^2 = -9$. With the method: $x^2 - y^2 = -9$ and $2xy = 0$ give $x = 0$, $y = \pm 3$.

Example 5 • Square roots

Find the square roots of $7 + 24i$.

$$\pm(4 + 3i)$$

Let $(x + iy)^2 = 7 + 24i$. Then $x^2 - y^2 = 7$ (1) and $2xy = 24$, so $y = \frac{12}{x}$ (2). Substitute (2) into (1): $x^2 - \frac{144}{x^2} = 7$, so $x^4 - 7x^2 - 144 = 0$, that is $(x^2 - 16)(x^2 + 9) = 0$. Since x is real, $x^2 = 16$, so $x = \pm 4$ and $y = \pm 3$ (same sign). Check: $(4 + 3i)^2 = 16 - 9 + 24i$.

Example 6

(a) $-5 + 12i$

$$\pm(2 + 3i)$$

$x^2 - y^2 = -5$ and $x^2 + y^2 = 13$, so $x^2 = 4$, $y^2 = 9$. Since $b = 12 > 0$ the signs match: $\pm(2 + 3i)$.

Example 6 • A faster way

(b) $-i$

$$\pm \frac{1}{\sqrt{2}}(1 - i)$$

$x^2 - y^2 = 0$ and $x^2 + y^2 = 1$, so $x^2 = y^2 = \frac{1}{2}$. Since $b = -1 < 0$ the signs are opposite:

$$\pm \frac{1}{\sqrt{2}}(1 - i). \text{ Check: } (1 - i)^2 = -2i, \text{ and } \frac{1}{2}(-2i) = -i.$$

HSC-style • Complex coefficients

Solve $z^2 + 3z + (3 - i) = 0$.

$$z = -1 + i \text{ or } z = -2 - i.$$

$\Delta = 9 - 4(3 - i) = -3 + 4i$. For the square roots: $x^2 - y^2 = -3$ and $x^2 + y^2 = 5$ give $x^2 = 1$, $y^2 = 4$, same sign, so $\pm(1 + 2i)$. Then $z = \frac{-3 \pm (1 + 2i)}{2}$. Check: the sum is -3 and the product is $(-1 + i)(-2 - i) = 2 + i - 2i + 1 = 3 - i$.

Example 7 • A harder one

Solve $z^2 + 2(1 - i\sqrt{3})z + 2 + 2i\sqrt{3} = 0$.

$$z = (-1 + \sqrt{2}) + (\sqrt{3} - \sqrt{6})i \text{ or } z = (-1 - \sqrt{2}) + (\sqrt{3} + \sqrt{6})i.$$

$\Delta = 4(1 - i\sqrt{3})^2 - 4(2 + 2i\sqrt{3}) = 4(-2 - 2\sqrt{3}i) - 8 - 8\sqrt{3}i = -16 - 16\sqrt{3}i = 16(-1 - \sqrt{3}i)$. Square roots of $-1 - \sqrt{3}i$: $x^2 - y^2 = -1$ and $x^2 + y^2 = 2$ give $x^2 = \frac{1}{2}$, $y^2 = \frac{3}{2}$; $b < 0$ so the signs are opposite: $\pm\left(\frac{1}{\sqrt{2}} - \sqrt{\frac{3}{2}}i\right)$. So $\sqrt{\Delta} = \pm(2\sqrt{2} - 2\sqrt{6}i)$ and $z = -1 + \sqrt{3}i \pm (\sqrt{2} - \sqrt{6}i)$. Never leave a complex number under a square root sign: find its square roots first.

HSC-style • Using square roots

(i) Find the square roots of $-i$. (ii) Hence solve $z^2 + 2z + (1 + i) = 0$.

$$\text{(i)} \pm \frac{1}{\sqrt{2}}(1 - i). \text{ (ii)} z = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \text{ or } z = -1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

(i) is Example 6(b). (ii) $\Delta = 4 - 4(1 + i) = -4i$, so $\sqrt{\Delta} = \pm 2 \cdot \frac{1}{\sqrt{2}}(1 - i) = \pm\sqrt{2}(1 - i)$ and $z = \frac{-2 \pm \sqrt{2}(1 - i)}{2} = -1 \pm \frac{1}{\sqrt{2}}(1 - i)$. Completing the square gives the same: $(z + 1)^2 = -i$.

Exit ticket

1. Find the square roots of $-3 - 4i$.

$$\pm(1 - 2i)$$

$x^2 - y^2 = -3$ and $x^2 + y^2 = 5$, so $x^2 = 1$, $y^2 = 4$. Since $b = -4 < 0$ the signs are opposite.

Check: $(1 - 2i)^2 = 1 - 4i - 4 = -3 - 4i$.

2. Hence solve $z^2 - 3z + (3 + i) = 0$.

$$z = 2 - i \text{ or } z = 1 + i.$$

$\Delta = 9 - 4(3 + i) = -3 - 4i$, so $\sqrt{\Delta} = \pm(1 - 2i)$ and $z = \frac{3 \pm (1 - 2i)}{2}$. Check: the sum is 3 and the product is $(2 - i)(1 + i) = 3 + i$.

Practice worksheet

Real coefficients

1. Solve.

(a) $z^2 + 6z + 13 = 0$

(b) $2z^2 - 2z + 5 = 0$

(a) $(z + 3)^2 = -4$, so $z = -3 \pm 2i$. **(b)** $z = \frac{2 \pm \sqrt{4-40}}{4} = \frac{2 \pm 6i}{4} = \frac{1}{2} \pm \frac{3}{2}i$.

2. Find a quadratic equation with real coefficients that has $-1 + 3i$ as a root.

The other root is $-1 - 3i$. Sum = -2 , product = $1 + 9 = 10$. So $z^2 + 2z + 10 = 0$.

3. The equation $z^2 + pz + q = 0$, where $p, q \in \mathbb{R}$, has $1 - 2i$ as a root. Find p and q .

The other root is $1 + 2i$. Sum = $2 = -p$, so $p = -2$. Product = $1 + 4 = 5 = q$.

Square roots

4. Find the square roots of (a) -16 , (b) $5 - 12i$, (c) $8 + 6i$.

(a) $\pm 4i$. **(b)** $x^2 - y^2 = 5, x^2 + y^2 = 13$, so $x^2 = 9, y^2 = 4$, **opposite signs:** $\pm(3 - 2i)$. **(c)** $x^2 - y^2 = 8, x^2 + y^2 = 10$, so $x^2 = 9, y^2 = 1$, **same signs:** $\pm(3 + i)$.

5. Find the square roots of $1 + i\sqrt{3}$ in exact form.

$x^2 - y^2 = 1$ and $x^2 + y^2 = 2$, so $x^2 = \frac{3}{2}$ and $y^2 = \frac{1}{2}$. Since $b > 0$ the signs match:
 $\pm \left(\sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i \right) = \pm \frac{\sqrt{6} + \sqrt{2}i}{2}$.

Complex coefficients

6. Solve $z^2 - (5 + i)z + (8 + i) = 0$.

$\Delta = (5 + i)^2 - 4(8 + i) = 24 + 10i - 32 - 4i = -8 + 6i$. **Square roots:** $x^2 - y^2 = -8, x^2 + y^2 = 10$, so $x^2 = 1, y^2 = 9$, **same signs:** $\pm(1 + 3i)$. Then $z = \frac{5+i \pm (1+3i)}{2}$, so $z = 3 + 2i$ or $z = 2 - i$.

HSC-style

7. (a) Find the square roots of $5 + 12i$. (b) Hence solve $z^2 - 2z - 4 - 12i = 0$.

(a) $x^2 - y^2 = 5$ and $x^2 + y^2 = 13$, so $x^2 = 9, y^2 = 4$, **same signs:** $\pm(3 + 2i)$. **(b)** $(z - 1)^2 = 5 + 12i$, so $z - 1 = \pm(3 + 2i)$ and $z = 4 + 2i$ or $z = -2 - 2i$. **Check: the sum is 2 and the product is $(4 + 2i)(-2 - 2i) = -4 - 12i$.**