

# Worked solutions

Lesson 2 • Conjugates, division and equality

## Warm-up • Real products

(a)  $(5 + i)(5 - i)$

$$26$$

$25 - i^2 = 25 + 1 = 26$ . The middle terms cancel.

(b)  $(a + ib)(a - ib)$

$$a^2 + b^2$$

The same pattern as (a):  $a^2 - i^2b^2 = a^2 + b^2$ , which is always real.

(c) Rationalise the denominator of  $\frac{1}{3 - \sqrt{2}}$ .

$$\frac{3 + \sqrt{2}}{7}$$

Multiply top and bottom by  $3 + \sqrt{2}$ :  $\frac{3+\sqrt{2}}{9-2}$ . This is the template for dividing complex numbers.

## Example 1

(a)  $\overline{-1 + 4i}$

$$-1 - 4i$$

Change the sign of the imaginary part.

(b)  $\overline{-2i}$

$$2i$$

$-2i = 0 - 2i$ , so its conjugate is  $0 + 2i$ .

(c) If  $z = 3 + 4i$ , find  $z\bar{z}$  and  $|z|$ .

$$z\bar{z} = 25 \text{ and } |z| = 5.$$

$$(3 + 4i)(3 - 4i) = 9 + 16 = 25, \text{ and } |z| = \sqrt{25} = 5.$$

(d)  $\overline{\overline{2 + 3i}}$

$$2 + 3i$$

Conjugating twice returns the original number.

## Example 2 • Proving the conjugate rules

- (a) Prove that  $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ .

$$z_1 + z_2 = (a + c) + i(b + d)$$

$$\overline{z_1 + z_2} = (a + c) - i(b + d) = (a - ib) + (c - id) = \overline{z_1} + \overline{z_2}$$

Add first, then conjugate; the result matches conjugating first, then adding.

- (b) Prove that  $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$ .

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

$$\overline{z_1 z_2} = (ac - bd) - i(ad + bc)$$

$$\overline{z_1} \overline{z_2} = (a - ib)(c - id) = ac - iad - ibc + i^2 bd = (ac - bd) - i(ad + bc)$$

**The two sides are equal.**

Expand both sides and compare real and imaginary parts. These identities are part of the syllabus: you must be able to prove and use them.

## Example 3 • Using the conjugate rules

- (a)  $\overline{(5 + i) + (3 - 2i)}$

$$8 + i$$

Directly:  $(8 - i)$  conjugated is  $8 + i$ . By the rule:  $(5 - i) + (3 + 2i) = 8 + i$ .

- (b)  $\overline{(5 + i)(3 - 2i)}$

$$17 + 7i$$

By the rule:  $(5 - i)(3 + 2i) = 15 + 10i - 3i + 2 = 17 + 7i$ . Directly:  $(5 + i)(3 - 2i) = 17 - 7i$ , whose conjugate is  $17 + 7i$ .

## Example 4

- (a)  $\frac{3}{2i}$

$$-\frac{3}{2}i$$

Multiply by  $\frac{i}{i}$ :  $\frac{3i}{2i^2} = -\frac{3i}{2}$ . Multiplying by  $\frac{-2i}{-2i}$  works too, but is slower.

- (b)  $\frac{1}{3 + 2i}$

$$\frac{3}{13} - \frac{2}{13}i$$

$$\frac{1}{3+2i} \times \frac{3-2i}{3-2i} = \frac{3-2i}{9+4}$$

### Example 5 • Division

Express  $\frac{3+4i}{2-3i}$  in the form  $a+ib$ .

$$-\frac{6}{13} + \frac{17}{13}i$$

$$\frac{(3+4i)(2+3i)}{(2-3i)(2+3i)} = \frac{6+9i+8i-12}{4+9} = \frac{-6+17i}{13}$$

Check the answer on a calculator in complex mode (the syllabus says “with and without digital tools”).

### Example 6

(a) Prove each relationship.

**Let  $z = a+ib$ . Then  $z + \bar{z} = 2a$ , so  $a = \frac{z+\bar{z}}{2}$ . Also  $z - \bar{z} = 2ib$ , so  $b = \frac{z-\bar{z}}{2i}$ . Finally  $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$ .**  
Add and subtract  $z$  and its conjugate, then use  $z\bar{z} = |z|^2$ .

(b) Hence show that  $z$  is real if and only if  $z = \bar{z}$ .

$$z \text{ is real} \Leftrightarrow \text{Im}(z) = 0 \Leftrightarrow \frac{z-\bar{z}}{2i} = 0 \Leftrightarrow z = \bar{z}.$$

Every step is reversible, so the argument proves both directions.

(c) What can you say about  $z \neq 0$  if  $\bar{z} = -z$ ?

**$z$  is purely imaginary.**

$$\bar{z} = -z \Leftrightarrow z + \bar{z} = 0 \Leftrightarrow 2\text{Re}(z) = 0 \Leftrightarrow \text{Re}(z) = 0.$$

### Example 7

Find real  $a$  and  $b$  such that  $z = a+ib$  satisfies  $z + 3i\bar{z} = -1 + 13i$ .

$$a = 5, b = -2, \text{ so } z = 5 - 2i.$$

$z + 3i\bar{z} = (a+ib) + 3i(a-ib) = (a+3b) + i(3a+b)$ . Real parts:  $a+3b = -1$ . Imaginary parts:  $3a+b = 13$ . Eliminating:  $3 \times (\text{first}) - (\text{second})$  gives  $8b = -16$ , so  $b = -2$  and  $a = 5$ .

$$\text{Check: } 5 - 2i + 3i(5 + 2i) = 5 - 2i + 15i - 6 = -1 + 13i.$$

### HSC practice • Conjugates and division

Let  $z = 5+i$  and  $w = 2-4i$ . Express  $\frac{\bar{z}}{w}$  in the form  $x+iy$ .

$$\frac{7}{10} + \frac{9}{10}i$$

$$\frac{5-i}{2-4i} \times \frac{2+4i}{2+4i} = \frac{10+20i-2i+4}{20} = \frac{14+18i}{20}.$$

## HSC practice • Conjugates and division

Let  $z = 1 + 3i$  and  $w = 2 - i$ . Find, in the form  $x + iy$ : (i)  $\bar{z}w$ ; (ii)  $\frac{z}{w}$ .

(i)  $-1 - 7i$ . (ii)  $-\frac{1}{5} + \frac{7}{5}i$ .

(i)  $(1 - 3i)(2 - i) = 2 - i - 6i + 3i^2 = -1 - 7i$ . (ii)  $\frac{(1+3i)(2+i)}{5} = \frac{2+i+6i-3}{5} = \frac{-1+7i}{5}$ .

## HSC practice • Equating parts

Let  $z = 2 + 3i$  and  $w = 1 - 5i$ . Find in the form  $x + iy$ : (i)  $z + \bar{w}$ ; (ii)  $z^2$ ; (iii)  $\frac{z}{w}$ .

(i)  $3 + 8i$ . (ii)  $-5 + 12i$ . (iii)  $-\frac{1}{2} + \frac{1}{2}i$ .

(i)  $\bar{w} = 1 + 5i$ , so  $z + \bar{w} = 3 + 8i$ . (ii)  $z^2 = 4 + 12i + 9i^2 = -5 + 12i$ . (iii)  $\frac{(2+3i)(1+5i)}{|w|^2} = \frac{2+10i+3i-15}{26} = \frac{-13+13i}{26}$ .

## HSC practice • Equating parts

Find the complex number  $z$  such that  $|z| = z + 8 + 12i$ .

$z = 5 - 12i$

$|z|$  is real, so the imaginary part of the right-hand side must be zero. Let  $z = a + ib$ : then  $b + 12 = 0$ , so  $b = -12$ . The real parts give  $\sqrt{a^2 + 144} = a + 8$ , so  $a^2 + 144 = a^2 + 16a + 64$  and  $a = 5$ . Check:  $a + 8 = 13 > 0$  and  $|5 - 12i| = 13$ .

## Exit ticket

1. Express  $\frac{5}{1 - 2i}$  in the form  $a + ib$ .

$\frac{1 + 2i}{(1-2i)(1+2i)} = \frac{5(1+2i)}{5} = 1 + 2i$ .

2. Find real  $x$  and  $y$  such that  $(x + iy)(2 - 3i) = -13i$ .

$x = 3, y = -2$

Expanding:  $(2x + 3y) + i(2y - 3x) = -13i$ , so  $2x + 3y = 0$  and  $2y - 3x = -13$ , giving  $x = 3$ ,  $y = -2$ . Alternatively divide:  $x + iy = \frac{-13i(2+3i)}{13} = -i(2 + 3i) = 3 - 2i$ .

## Practice worksheet

### Conjugates and modulus

1. Find  $\bar{z}$ ,  $z\bar{z}$  and  $|z|$  for (a)  $z = 5 - 12i$ , (b)  $z = -\sqrt{3} + i$ .

(a)  $\bar{z} = 5 + 12i$ ,  $z\bar{z} = 25 + 144 = 169$ ,  $|z| = 13$ . (b)  $\bar{z} = -\sqrt{3} - i$ ,  $z\bar{z} = 3 + 1 = 4$ ,  $|z| = 2$ .

2. Let  $z_1 = 2 - i$  and  $z_2 = 4 + 3i$ . Verify that  $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ .

$z_1 z_2 = 8 + 6i - 4i - 3i^2 = 11 + 2i$ , so  $\overline{z_1 z_2} = 11 - 2i$ . Also  $\bar{z}_1 \bar{z}_2 = (2 + i)(4 - 3i) = 8 - 6i + 4i + 3 = 11 - 2i$ . They agree.

3. Prove that  $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ .

**Let  $z_1 = a + ib$  and  $z_2 = c + id$ . Then  $z_1 - z_2 = (a - c) + i(b - d)$ , so  $\overline{z_1 - z_2} = (a - c) - i(b - d) = (a - ib) - (c - id) = \overline{z_1} - \overline{z_2}$ .**

### Division

4. Express in the form  $a + ib$ .

(a)  $\frac{4}{i}$

(b)  $\frac{2}{1+i}$

(c)  $\frac{1+2i}{3-i}$

(d)  $\frac{5+i}{2+3i}$

**(a)  $\frac{4i}{i^2} = -4i$ . (b)  $\frac{2(1-i)}{2} = 1 - i$ . (c)  $\frac{(1+2i)(3+i)}{10} = \frac{3+i+6i-2}{10} = \frac{1+7i}{10} = \frac{1}{10} + \frac{7}{10}i$ . (d)  $\frac{(5+i)(2-3i)}{13} = \frac{10-15i+2i+3}{13} = 1 - i$ .**

5. Use  $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$  to find  $\frac{1}{z}$  when  $z = -3 + 4i$ .

**$|z|^2 = 25$  and  $\bar{z} = -3 - 4i$ , so  $\frac{1}{z} = \frac{-3-4i}{25} = -\frac{3}{25} - \frac{4}{25}i$ .**

### Equality of complex numbers

6. Find real  $x$  and  $y$  such that  $(x + iy)(1 + 2i) = 7 + 4i$ .

**$(x - 2y) + i(2x + y) = 7 + 4i$ , so  $x - 2y = 7$  and  $2x + y = 4$ . Solving:  $x = 3, y = -2$ .**

7. Show that  $z = \bar{z}$  if and only if  $z$  is real.

**Let  $z = a + ib$ . Then  $z = \bar{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow 2ib = 0 \Leftrightarrow b = 0 \Leftrightarrow z$  is real.**

### Adapted HSC-style

8. Let  $z = 3 - 2i$  and  $w = 1 + 4i$ . (a) Express  $\frac{w}{z}$  in the form  $x + iy$ . (b) Find real  $a$  and  $b$  such that  $z + (a + ib)\bar{w} = 10 + i$ .

**(a)  $\frac{(1+4i)(3+2i)}{13} = \frac{3+2i+12i-8}{13} = -\frac{5}{13} + \frac{14}{13}i$ . (b)  $(a + ib)(1 - 4i) = (a + 4b) + i(b - 4a)$ , so  $3 + a + 4b = 10$  and  $-2 + b - 4a = 1$ , that is  $a + 4b = 7$  and  $b - 4a = 3$ . Then  $b = 4a + 3$  gives  $17a + 12 = 7$ , so  $a = -\frac{5}{17}, b = \frac{31}{17}$ .**