

Worked solutions

Lesson 1 • The arithmetic of complex numbers

Warm-up • Why do we need a new number?

- (a) Find the sum and the product of the roots.

Sum = -2 , product = 4 .

For $ax^2 + bx + c = 0$ the sum of the roots is $-\frac{b}{a}$ and the product is $\frac{c}{a}$. These formulae still make sense even if the roots are not real.

- (b) Use the quadratic formula. What goes wrong?

$x = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm \sqrt{-12}}{2}$. **The discriminant is $\Delta = -12 < 0$, so there are no real roots.**

We need a number whose square is negative. Once we have one, the roots are $x = -1 \pm i\sqrt{3}$.

Check: the sum is -2 and the product is $1 + 3 = 4$.

Discuss

If multiplying by a number i twice has the same effect as multiplying by -1 , what must i^2 equal? What does multiplying by i do to the number line?

$i^2 = -1$. Multiplying by i is a quarter-turn (90°): it takes 1 off the real number line to a new vertical axis.

This is only a preview. In Lesson 4 we plot complex numbers on the Argand diagram, and in Lesson 8 we use multiplication by i as a rotation.

Example 1 • Solve

- (a) $x^2 + 9 = 0$

$$x = \pm 3i$$

$$x^2 = -9 = 9 \times (-1), \text{ so } x = \pm 3i.$$

- (b) $x^2 + 5 = 0$

$$x = \pm i\sqrt{5}$$

$$x^2 = -5, \text{ so } x = \pm \sqrt{5}i.$$

- (c) $4x^2 + 49 = 0$

$$x = \pm \frac{7i}{2}$$

$$x^2 = -\frac{49}{4}, \text{ so } x = \pm \frac{7}{2}i.$$

Example 2 • Powers of i

(a) i^2

$$-1$$

This is the definition of i .

(b) i^3

$$-i$$

$$i^3 = i^2 \times i = -i.$$

(c) i^4

$$1$$

$$i^4 = (i^2)^2 = (-1)^2 = 1.$$

(d) i^5

$$i$$

$$i^5 = i^4 \times i = i.$$

(e) i^6

$$-1$$

$$i^6 = i^4 \times i^2 = -1.$$

(f) i^7

$$-i$$

$$i^7 = i^4 \times i^3 = -i. \text{ Can you spot the cycle of four before the next slide?}$$

(g) i^8

$$1$$

$$i^8 = (i^4)^2 = 1.$$

(h) i^{40}

$$1$$

$$i^{40} = (i^4)^{10} = 1^{10} = 1.$$

(i) i^{50}

$$-1$$

$$i^{50} = (i^4)^{12} \times i^2 = -1.$$

(j) i^{231}

$$-i$$

$$231 = 4 \times 57 + 3, \text{ so } i^{231} = (i^4)^{57} \times i^3 = -i.$$

HSC question

Evaluate $i + i^2 + i^3 + i^4 + i^5$.

i

$i + (-1) + (-i) + 1 + i = i$. Any four consecutive powers of i add to 0, so only the fifth term survives. Extension: $i + i^2 + \dots + i^{2027} = -1$, since $2027 = 4 \times 506 + 3$ leaves $i + i^2 + i^3 = -1$.

Example 3 • Real and imaginary parts

(a) $3 + 4i$

$\text{Re} = 3, \text{Im} = 4$

$\text{Im}(z)$ is 4, not $4i$.

(b) $-2i$

$\text{Re} = 0, \text{Im} = -2$

$-2i = 0 + (-2)i$, so z is purely imaginary.

(c) 7

$\text{Re} = 7, \text{Im} = 0$

$7 = 7 + 0i$. Every real number is a complex number.

(d) $\frac{1 - i\sqrt{5}}{2}$

$\text{Re} = \frac{1}{2}, \text{Im} = -\frac{\sqrt{5}}{2}$

Split it as $\frac{1}{2} - \frac{\sqrt{5}}{2}i$.

Example 4 • Classifying numbers

(a) $5, \frac{18}{6}$

$5 \in \mathbb{N}$ and $\frac{18}{6} = 3 \in \mathbb{N}$

Whole numbers 1, 2, 3, ... (some texts include 0 in $\mathbb{N}$; the syllabus does not insist either way).

(b) -8

$\mathbb{Z}$

A negative whole number is an integer but not a natural number.

(c) $\sin \frac{\pi}{6}, \frac{22}{7}$

$\sin \frac{\pi}{6} = \frac{1}{2} \in \mathbb{Q}$ and $\frac{22}{7} \in \mathbb{Q}$

Both are ratios of integers. Note $\frac{22}{7}$ is only an approximation of π .

(d) $\pi, \sqrt{5}$

$\mathbb{R}$ (irrational)

They are real but cannot be written as a ratio of integers.

(e) $\sqrt{49}$

$7 \in \mathbb{N}$

$\sqrt{49} = 7$ exactly, so it is a natural number.

(f) $2 + 3i, \quad 4i$

$\mathbb{C}$

Both are non-real, so $\mathbb{C}$ is the smallest set containing them.

Example 5

(a) $(3 + 4i) + (2 + 6i)$

$5 + 10i$

Add real parts and add imaginary parts.

(b) $(4 - 3i) - (2 - 4i)$

$2 + i$

$(4 - 2) + (-3 + 4)i = 2 + i$. Take care with the signs when subtracting.

(c) $(3 + 2i)(4 + 2i)$

$8 + 14i$

$12 + 6i + 8i + 4i^2 = 12 + 14i - 4 = 8 + 14i$.

Example 6 • Multiplying

(a) $(2 + 5i)(3 + 6i)$

$-24 + 27i$

$6 + 12i + 15i + 30i^2 = 6 + 27i - 30 = -24 + 27i$.

(b) $(3 + 2i)(3 - 2i)$

13

$9 - 6i + 6i - 4i^2 = 9 + 4 = 13$. The middle terms cancel.

(c) $(a + ib)(a - ib)$

$a^2 + b^2$

The product is always real. The numbers $a + ib$ and $a - ib$ are called **conjugates** (Lesson 2).

HSC practice • Expanding a power

Expand $(a + ib)^3$, where $a, b \in \mathbb{R}$, and write it in the form $x + iy$. Hence find $(2 - i)^3$.

$(a + ib)^3 = (a^3 - 3ab^2) + i(3a^2b - b^3)$, and $(2 - i)^3 = 2 - 11i$.

$(a + ib)^3 = a^3 + 3a^2(ib) + 3a(ib)^2 + (ib)^3 = a^3 + 3a^2bi - 3ab^2 - b^3i$, using $i^2 = -1$ and $i^3 = -i$. For $(2 - i)^3$ put $a = 2, b = -1$: real part $8 - 6 = 2$, imaginary part $-12 + 1 = -11$. A common slip is forgetting that $i^3 = -i$.

Exit ticket

1. Simplify i^{2027} .

$-i$

$2027 = 4 \times 506 + 3$, so $i^{2027} = (i^4)^{506} \times i^3 = -i$.

2. Express $(1 + i)^4$ in the form $a + ib$.

-4

$(1 + i)^2 = 1 + 2i + i^2 = 2i$, so $(1 + i)^4 = (2i)^2 = 4i^2 = -4$.

3. Find $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ if $z = (2 - i)(3 + 4i)$.

$\operatorname{Re}(z) = 10, \operatorname{Im}(z) = 5$

$z = 6 + 8i - 3i - 4i^2 = 6 + 5i + 4 = 10 + 5i$.

Practice worksheet

The number i and powers of i

1. Solve each equation.

(a) $x^2 + 16 = 0$

(b) $x^2 + 7 = 0$

(c) $9x^2 + 4 = 0$

(a) $x = \pm 4i$. **(b)** $x = \pm i\sqrt{7}$. **(c)** $x^2 = -\frac{4}{9}$, so $x = \pm \frac{2i}{3}$.

2. Simplify.

(a) i^{12}

(b) i^{27}

(c) i^{102}

(d) i^{-1}

(a) $i^{12} = (i^4)^3 = 1$. **(b)** $27 = 4 \times 6 + 3$, so $i^{27} = -i$. **(c)** $102 = 4 \times 25 + 2$, so $i^{102} = -1$. **(d)** $i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$.

Number sets and parts

3. State the smallest set ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) containing each number.

(a) $-\frac{3}{4}$

(b) $\sqrt{16}$

(c) $\sqrt{12}$

(d) $5i - 2$

(e) $\cos \pi$

(a) $\mathbb{Q}$. **(b)** $\sqrt{16} = 4$, so $\mathbb{N}$. **(c)** $\mathbb{R}$ (irrational). **(d)** $\mathbb{C}$. **(e)** $\cos \pi = -1$, so $\mathbb{Z}$.

4. Write down $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ for (a) $z = -5 + 2i$, (b) $z = 6i$, (c) $z = \frac{3-i}{4}$.

(a) $\operatorname{Re} = -5, \operatorname{Im} = 2$. **(b)** $\operatorname{Re} = 0, \operatorname{Im} = 6$. **(c)** $z = \frac{3}{4} - \frac{1}{4}i$: $\operatorname{Re} = \frac{3}{4}, \operatorname{Im} = -\frac{1}{4}$.

Arithmetic

5. Express in the form $a + ib$.

(a) $(5 - 2i) + (1 + 7i)$

(b) $(2 + 3i) - (6 - i)$

(c) $(4 + i)(2 - 3i)$

(d) $(1 - 2i)^2$

(a) $6 + 5i$. **(b)** $-4 + 4i$. **(c)** $8 - 12i + 2i - 3i^2 = 11 - 10i$. **(d)** $1 - 4i + 4i^2 = -3 - 4i$.

6. Find $\text{Re}(z)$ and $\text{Im}(z)$ if $z = (2 - i)(3 + 4i)$.

$z = 6 + 8i - 3i - 4i^2 = 10 + 5i$, so $\text{Re}(z) = 10$ and $\text{Im}(z) = 5$.

Adapted HSC-style

7. Let $z = 1 + i$. (a) Find z^2 and z^3 in the form $a + ib$. (b) Hence evaluate $z + z^2 + z^3$. (c) Find z^8 .

(a) $z^2 = 1 + 2i + i^2 = 2i$ and $z^3 = 2i(1 + i) = 2i - 2 = -2 + 2i$. **(b)** $(1 + i) + 2i + (-2 + 2i) = -1 + 5i$. **(c)** $z^8 = (z^2)^4 = (2i)^4 = 16i^4 = 16$.